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Some Applications of the Spectral of Iterated Heterologous Bi-subdivision Graphs

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Abstract: Among many graph operations, many researchers have studied the subdivision of a graph because of its beautiful graph structure. Some scholars put forward subdivided-line graph operation to construct extended Sierpiński graphs. Notice that the middle graph put the line graph operation and subdivision graph operation together. Take into account these studies, we introduce the heterologous bi-subdivision graph operation. Depending on this graph operation, we construct iterated heterologous bi-subdivision graphs. Then, we study some applications of Laplacian eigenvalues of iterated models, for example second smallest and largest eigenvalues, resistance distance, first- and second-order network coherence.

Keywords: Bi-subdivision graphs, Laplacian eigenvalues, algebraic connectivity, spectral radius, resistance distance, coherence

1 Introduction

In the past decade, graph theory has gone through a remarkable shift and profound transformation. Knowledge of graph's Laplacian spectra is central to understanding its structure and dynamics, efforts have been undertaken to study the Laplacian spectra of graphs. Spectral graph theory has become a heated subject of extensive empirical research in various scientific fields, such as physics, mathematics [1, 2], computer science [3], due to its wide applications in this academic field [4–6].

In the last few years, many scholars use spectral graph theory to analyze dynamical and structural properties of graphs [7, 8], for example the trapping problem[9, 10], Kirchhoff index[11], Eigentime identities[12, 13] and the robustness of consensus algorithm with Gaussian white noise[14–18]. Among various graphs, Sierpińshi graphs are one of the most important categories[19–21]. T. Hasunuma newly introduced the subdivided-line graph operation and studied the structural properties of subdivided-line graphs [22]. Z. Zhang and Y. Qi studied the spectral properties of extended Sierpiński graphs which constructed by doing subdivided-line graph operation to a complete graph [23]. The subdivided-line graph of a graph combined two graph operations which are the line graph of a graph [24] and the subdivision of a graph [25, 26]. A. Aytaç studied some properties of middle graph of graphs [27] which put the line graph and subdivision graph together.

The concept of a line graph, that transforms links of the original graph into nodes in the line graph, can be used to understand the influence of link failures on infrastructure networks. In the past, some researchers have studied the Laplacian spectrum of bi-subdivision. On this basis, we combine line graph with bi-subdivision graph. Based on these studies, we first introduce heterologous bi-subdivision graph operation. That is, we first construct the bi-subdivision graph of graph. And, we divide the new nodes of bi-subdivision graph into two categories. Then, we link some nodes of the same category. New vertices of bi-subdivision graph are heterologous if created by the different edge. That's why we divide the nodes into two categories and the model call heterologous bi-subdivision graph. According to this graph operation, we construct iterated heterologous bi-subdivision graphs.

The organization of this paper is as follows. In Sec.2, we introduced the heterologous bi-subdivision graph operation and construct iterated heterologous bi-subdivision graphs. In Sec.3, we study the iteration formula of Laplacian eigenvalues of iterated heterologous bi-subdivision graphs when the initial graph is regular. In Sec.4, we give some application of Laplacian eigenvalues of iterated models. In the last section, we draw the conclusions.

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2 Model

In order to construct our models, we first introduce some notions.

Let $G = (V, E)$ be a connected undirected graph with vertex set $V = V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \cdots, e_m\}$. The graph can be called as k-regular graph when their vertices all have the same degree k.

Let $d(v_i), i \in \{1, 2, \cdots, n\}$ denote the degree of vertex v_i. Let $D(G) = \text{diag}(d(v_1), d(v_2), \cdots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix is $L(G) = D(G) - A(G)$, where $A(G)$ is the $(0, 1)$-adjacency matrix.

The incidence matrix $R(G)$ of G a $n \times m$ matrix, whose $(i, j)^{th}$ entry is defined as follows:

1. 1 if vertex v_i is incident on edge e_j, (i.e., v_i is one of the endpoints of e_j.)
2. 0 if vertex v_i is not incident on edge e_j, (i.e., v_i isn't one of the endpoints of e_j.)

The line graph $Y(G)$ of a graph G is the graph whose vertex set is in one-to-one correspondence with the edge set $E(G)$ of G, where two vertices are adjacent if and only if their corresponding edges of G have a vertex in common.

The bi-subdivision graph, denoted by $B^2(G)$, is the graph obtained by inserting 2 paths of length 2 replacing the every edge of G. Here $B^2(G)$ is called the bi-subdivided operation.

The vertex set of $B^2(G)$ makes up of $n + 2m$ vertices. The old vertex set V makes up of n elements. The new vertex set $V^{new}(B^2(G)) = V_1^{new} \cup V_2^{new}$, where V_1^{new}=$\{v_{e_j}^1\}$ and V_2^{new}=$\{v_{e_j}^2\}$ consists of $2m$ elements together $(j = 1, 2, \cdots, m)$. The $v_{e_j}^1$ and $v_{e_j}^2$ describe the different inserting vertices created by corresponding edges e_j. The relationship of new vertices created by different edges is called heterology. (Fig.1 illustrates the construction processes of $B^2(G)$ with new vertex set V_1^{new} and V_2^{new}.)

Now we can construct our model by using heterologous bi-subdivision graph operation.

The heterologous bi-subdivision graph $B_{he}^2(G)$ is constructed as follows: (Fig.1, we show the construction for $B_{he}^2(G)$ for the case of $n = 4$ and $k = 3$.)

1. The bi-subdivision graph of the graph G, $B^2(G)$, is the graph obtained by inserting 2 paths of length 2 replacing the every edge of G.
2. The two distinct inserting vertices of $B^2(G)$ belonging to the same heterologous subset V_i^{new} $(i = 1, 2)$ are adjacent if and only if their corresponding edges of G are adjacent.

Let $G_0 = (V_0, E_0)$ be k-regular graph with vertex set $V_0 = V(G_0) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E_0 = E(G_0) = \{e_1, e_2, \cdots, e_m\}$. It is obvious that $2m = nk$.

Now we introduce the iterated heterologous bi-subdivision graphs G_t, which are built in an iterative way.

1. The initial graph is k-regular graph G_0.
2. For $t \geq 1$, $G_t = B_{he}^2(G_{t-1})$.

According to the construction, we can obtain G_t is $2^t k$-regular graph. Let N_t and E_t denote the total number of nodes and the total number of edges, respectively. We can obtain the recursion relations of N_t and E_t. That is

$$N_t = 2E_{t-1} + N_{t-1},$$

$$E_t = \frac{2^t k N_t}{2}.$$

Then, we can obtain $N_t = (2^{t-1}k + 1)N_{t-1}$. Through this recursion relation, we have

$$N_t = \prod_{i=1}^{t} (2^{i-1}k + 1)n, \tag{1}$$

and

$$E_t = 2^{t-1}k \prod_{i=1}^{t} (2^{i-1}k + 1)n, \tag{2}$$

where n is the number of nodes of initial graph G_0.

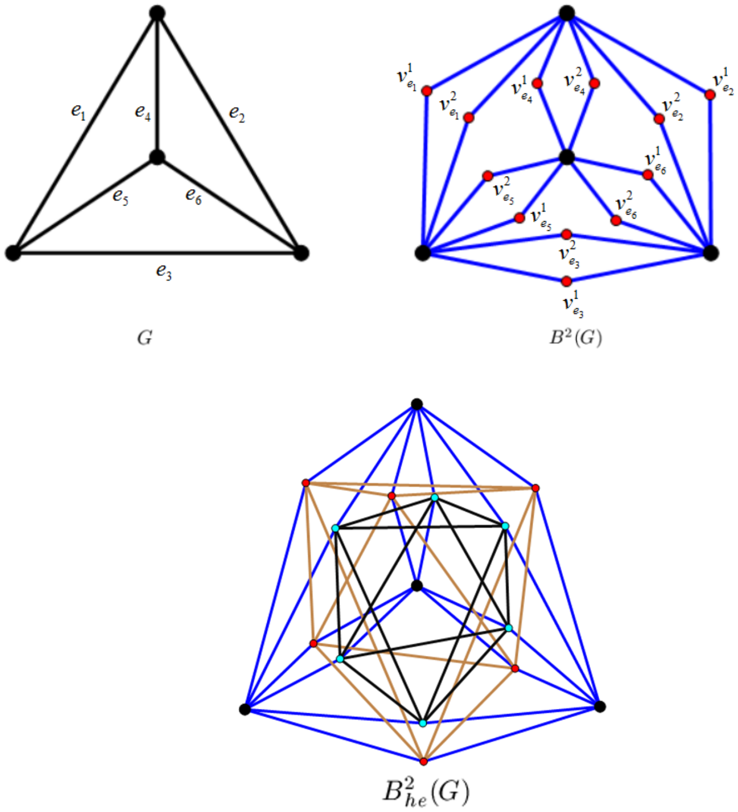

Figure 1: Illustrate of the construction for $B_{he}^2(G)$ for the case of $n = 4$ and $k = 3$.

3 Laplacian spectrum

To facilitate the calculation of the eigenvalues of Laplacian matrices, we introduce several Lemmas.

Lemma 3.1[29] Let G_0 be a k-regular graph with adjacency matrix A and an incidence matrix R. Let $Y(G)$ be line graph. Then $R^T R = A(Y(G)) + 2I$.

Lemma 3.2 [30] Let M_1, M_2, M_3, M_4 be respectively $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrix. If M_1 are invertible, then

$$\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2).$$

Let $P_{L(G)}(\lambda)$ denote the characteristic polynomial of matrix $L(G)$. Let $P_{A(Y(G))}(\lambda)$ denote the characteristic polynomial of matrix $A(G)$. That is $P_{L(G)}(\lambda) = det\big(\lambda I_m - L(G)\big)$ and $P_{A(Y(G))}(\lambda) = det\big(\lambda I_m - A(Y(G))\big)$.

Lemma 3.3 [31] If G is a k-regular graph with n vertices and m edges. We have the relationship between $P_{L(G)}(\lambda)$ and $P_{A(Y(G))}(\lambda)$,

$$P_{A(Y(G))}(\lambda) = (\lambda + 2)^{m-n} P_{L(G)}(\lambda + 2 - 2k).$$

We can easily derive lemma 3.4 based on the elementary transformation of matrix.

Lemma 3.4 The determinant of the bolck matrix of the form

$$\begin{pmatrix} A - B & -B \\ -B & A - B \end{pmatrix},$$

is $det(A)det(A - 2B)$, where the matrix A and B have same order.

Let L_t, D_t, A_t and R_t denote, respectively, the Laplacian matrix, degree matrix, adjacent matrix and incidence matrix of G_t. Let $A(Y(G_t))$ be the adjacent matrix of the line graph $Y(G_t)$.

Let $P_{L_t}(\lambda) = det(\lambda I_t - L_t)$ denote the characteristic polynomial of matrix L_t, where I_{N_t} is $N_t \times N_t$ identity matrix. The following proposition establishes the recursive relationship for the characteristic polynomial $P_t(\lambda)$.

Proposition 3.5 For $t \geq 1$, the characteristic polynomial $P_{L_t}(\lambda)$ of $L(G_t)$ satisfies the following relationship:

$$P_t(\lambda) = (2^t k - \lambda + 2)^{2E_{t-1} - N_{t-1}} P_{t-1}(\lambda - 2) P_{t-1}(\lambda).$$

Proof: Notice that, G_t is $2^t k$-regular graph and $G_t = B_{he}^2(G_{t-1})$. The characteristic polynomial $P_{L_t}(\lambda)$ can described in the following form.

$$P_{L_t}(\lambda) = \begin{vmatrix} (\lambda - 2^t k) I_{N_{t-1}} & R_{t-1} & R_{t-1} \\ R_{t-1}^T & (\lambda - 2^t k) I_{E_{t-1}} + A(Y(G_{t-1})) & 0 \\ R_{t-1}^T & 0 & (\lambda - 2^t k) I_{E_{t-1}} + A(Y(G_{t-1})) \end{vmatrix},$$

where $I_{N_{t-1}}$ and $I_{E_{t-1}}$ are N_{t-1} identity matrix and E_{t-1} identity matrix, respectively.

According to Lemma 3.1, we have $R_{t-1}^T R_{t-1} = A(Y(G_{t-1})) + 2I_{E_{t-1}}$. Then according to Lemma 3.4, we have

$$\begin{aligned} P_t(\lambda) &= (\lambda - 2^t k)^{N_{t-1}} det\Bigg[\begin{pmatrix} (\lambda - 2^t k) I_{E_{t-1}} + A(Y(G_{t-1})) & 0 \\ 0 & (\lambda - 2^t k) I_{E_{t-1}} + A(Y(G_{t-1})) \end{pmatrix} \\ &\quad - \frac{1}{\lambda - 2^t k} \begin{pmatrix} R_{t-1}^T \\ R_{t-1}^T \end{pmatrix} \begin{pmatrix} R_{t-1}, & R_{t-1} \end{pmatrix}\Bigg] \\ &= det\Bigg[\begin{pmatrix} (\lambda - 2^t k)^2 I_{E_{t-1}} + (\lambda - 2^t k) A(Y(G_{t-1})) & 0 \\ 0 & (\lambda - 2^t k)^2 I_{E_{t-1}} + (\lambda - 2^t k) A(Y(G_{t-1})) \end{pmatrix} \\ &\quad - \begin{pmatrix} R_{t-1}^T R_{t-1} & R_{t-1}^T R_{t-1} \\ R_{t-1}^T R_{t-1} & R_{t-1}^T R_{t-1} \end{pmatrix}\Bigg] (\lambda - 2^t k)^{N_{t-1} - 2E_{t-1}} \\ &= (\lambda - 2^t k)^{N_{t-1} - E_{t-1}} det\Big((\lambda - 2^t k) I_{E_{t-1}} + A(Y(G_{t-1}))\Big) \\ &\quad \times det\Big(((\lambda - 2^t k)^2 - 4) I_{E_{t-1}} + (\lambda - 2^t k - 2) A(Y(G_{t-1}))\Big) \\ &= (-1)^{E_{t-1}} (\lambda - 2^t k)^{N_{t-1} - E_{t-1}} det\Big((2^t k - \lambda) I_{E_{t-1}} - A(Y(G_{t-1}))\Big) \\ &\quad \times (\lambda - 2^t k - 2)^{E_{t-1}} det\Big(\frac{(2 - \lambda + 2^t k)(2 + \lambda - 2^t k)}{\lambda - 2^t k - 2} I_{E_{t-1}} - A(Y(G_{t-1}))\Big) \\ &= (-1)^{E_{t-1}} (\lambda - 2^t k)^{N_{t-1} - E_{t-1}} (2^t k - \lambda + 2)^{E_{t-1} - N_{t-1}} P_{t-1}(2 - \lambda) \\ &\quad \times (\lambda - 2^t k - 2)^{E_{t-1}} (2^t k - \lambda)^{E_{t-1} - N_{t-1}} P_{t-1}\Big(2^t k - 2 - \lambda + 2 - 2^t k\Big) \\ &= (2^t k - \lambda + 2)^{2E_{t-1} - N_{t-1}} P_{t-1}(\lambda - 2) P_{t-1}(\lambda). \quad \square \end{aligned}$$

According to Proposition 3.5, we can easily obtain the following Corollary.

Corollary 3.5.1 Let $\triangle_t = \{\lambda_1^{(t)}, \lambda_2^{(t)}, \cdots, \lambda_{N_t}^{(t)}\}$ denote the set of eigenvalues of Laplacian matrix L_t for G_t $(t \geq 1)$, where the distinctness of elements in $\triangle_t$ has been ignored and $0 = \lambda_1^{(t)} \leq \lambda_2^{(t)} \leq \cdots \leq \lambda_{N_t}^{(t)}$. Then

$$\triangle_t = \{2^t k + 2\}^{2E_{t-1} - N_{t-1}} \cup \{\triangle_{t-1} + 2\} \cup \{\triangle_{t-1}\},$$

where $\{\triangle_{t-1} + 2\}$ is $\{\lambda_1^{(t-1)} + 2, \lambda_2^{(t-1)} + 2, \cdots, \lambda_{N_{t-1}}^{(t-1)} + 2\}$.

According to Eqs.(1) and (2), we can obtain

$$2E_{t-1} - N_{t-1} = (2^{t-1} k - 1) \prod_{i=1}^{t-1} (2^{i-1} k + 1) n,$$

where G_0 is k-regular graph with n nodes.

Take G_0 be a complete graph with 4 nodes as an example, we have the eigenvalues of Laplacian matrix of G_0, that is {0, 4, 4, 4}. We know G_0 is 3-regular graph with m=6, n=4. In Fig.2, we show a histogram related to the distribution of eigenvalues of Laplacian matrix of G_3.

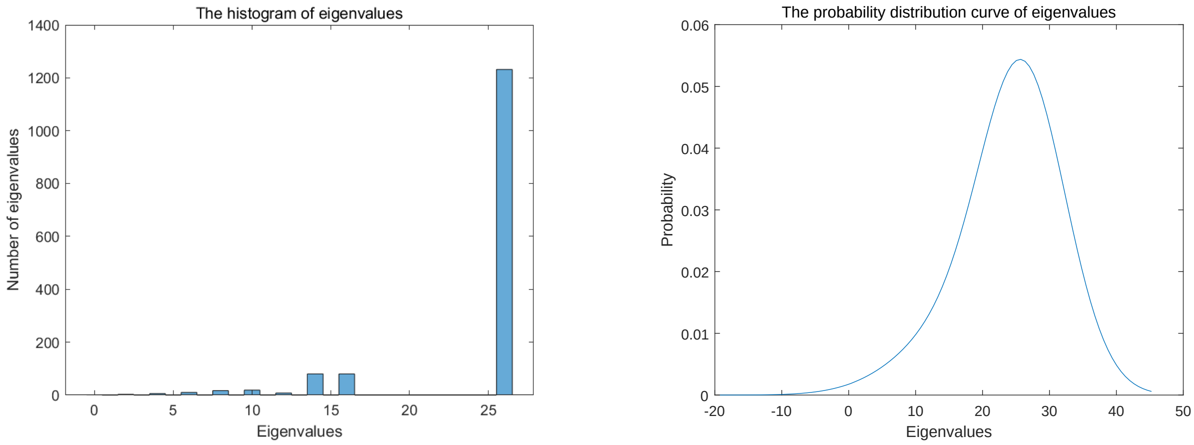

Figure 2: The histogram and probability distribution curve of Laplacian eigenvalues of G_3.

4 Application of Laplacian spectrum

The eigenvalues of the Laplacian matrix are important in graph theory. Especially, the largest and the second smallest eigenvalues of $L(G)$ are probably the most important information contained in the spectrum of a graph. We have know that the second smallest eigenvalue is called algebraic connectivity $a(G)$ which presented by Fiedler in 1973 [32]. The largest eigenvalue is called the Laplacian spectral radius of graph.

And, numerous studies have shown that the Laplacian eigenvalues of graph can characterize various structural and dynamical aspects of graph, including the number of spanning trees, the Kirchhoff index, mean hitting time and so on.

According to Corollary 3.5.1, we can obtain the Laplacian spectrum of iterated heterologous bi-subdivision graphs. Through accurate Laplacian spectrum, we study some basic variables of the iterated heterologous bi-subdivision graphs.

4.1 Second smallest eigenvalue

Among all elements in Laplacian spectrum, one of the most popular is the second smallest eigenvalue, called by Fiedler [32], the algebraic connectivity of a graph. Its importance is due to the fact that it is a useful parameter to measure, to a certain extent, how well a graph is connected. For example, it is well-known that a graph is connected if and only if its algebraic connectivity different from zero. In the past, the algebraic connectivity has received much more attention, see [33, 34] for surveys and books; [35–37] for application on trees; [38, 39] for applications on hard problems in graph theory.

In this paper, according to the construction of iterated heterologous bi-subdivision graphs G_t. Let λ_2^t denote the second smallest eigenvalues of Laplacian matrix of G_t. We have $\lambda_2^t = \lambda_2^{t-1} = \cdots = \lambda_2^1 = 2$.

The cover time of random walks on a graph is a fundamental observable to quantify the efficiency of exploring or searching the graph. It is intimately related to numerous combinatorial and algebraic properties of a graph, for example, conductance. Thus, evaluating or bounding the cover time can lead to some interesting combinatorial results and have attracted much attention [40, 41]. For a regular graph $G = (V, E)$ with $|V| = N$, Broder and Karlin [42] provided a bound for the cover time $C(G)$ of G given by $C(G) = \mathcal{O}(NlogN/(1-\sigma_2))$, where σ_2 is the second largest eigenvalue of transition probablity matrix of G.

Theorem 4.1.2 Let C_t denote the cover time of iterated heterologous bi-subdivision graphs G_t. Then, for large graphs,

$$C_t = \mathcal{O}\bigg(2^{t-1}kn\prod_{i=1}^{t}(2^{i-1}k+1)log(\prod_{i=1}^{t}(2^{i-1}k+1)n)\bigg),$$

where n are the number of nodes of initial graph G_0, G_0 is a k-regular graph.

Proof: Let σ_2^t denote the second largest eigenvalue of transition probablity matrix of G_t and λ_2^t denote the second smallest eigenvalues of Laplacian matrix of G_t. For a $2^t k$-regular graph [28], we can obtain the relation $1-\sigma_2^t = \lambda_2^t/2^t k$,

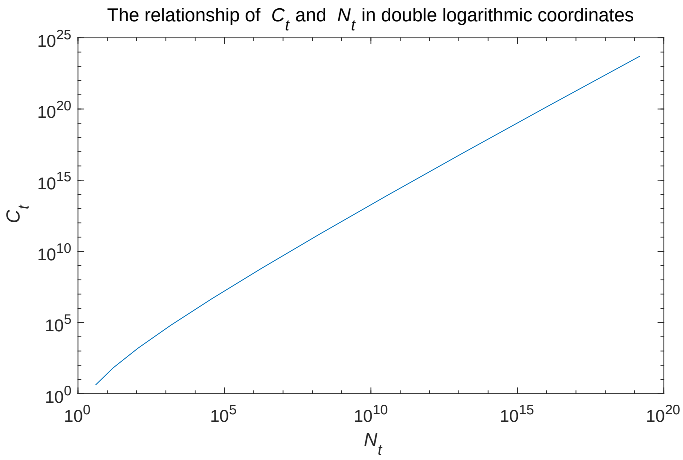

Figure 3: The relationship of C_t and N_t in double logarithmic coordinates for the case of G_0 is 3-regular graph with 4 nodes and 6 edges.

then the bound of the cover time of G_t can be expressed as

$$C_t = \mathcal{O}\left(2^t k \frac{N_t log N_t}{2}\right).$$

We have $N_t = \prod_{i=1}^{t}(2^{i-1}k+1)n$ where G_0 is k-regular graph with 4 nodes. And, we have $\lambda_2^t = 2$. □

In Fig.3, we plot the relationship of C_t and N_t in double logarithmic coordinates for the case of the initial graph G_0 is a complete graph with 4 nodes.

4.2 Largest eigenvalue

The largest eigenvalue of $L(G)$ is known as the the Laplacian spectral radius of the graph G. Let $\lambda_{N_t}^t$ denote the largest eigenvalue of $L(G_t)$. The importance of Laplacian spectral radius is attributed to the following two reasons. On the one hand, it is related to the algebraic connectivity of the complement of G [43] and some graph invariants [44, 45]. On the other hand, it is used in theoretical chemistry [52], combinatorial optimization [47] and communication networks [48] etc.

Theorem 4.2.1 Let $\lambda_{N_t}^t$ denote the largest eigenvalue of Laplacian matrix of iterated heterologous bi-subdivision graphs G_t. Then, for $n \geq 1$, we have

$$\lambda_{N_t}^t = 2^t k + 2,$$

where the initial graph is k-regular graph.

Proof: Let λ_n^0 denote the largest Laplacian eigenvalue of initial G_0, where G_0 is k-regular graph with n nodes. It is a well-known fact that $\lambda_n^0 \leq 2k$ [49]. According to Corollary 3.5.1, $\lambda_{N_1}^1$ is depends by $\lambda_n^0 + 2$ or $2k + 2$. We have $\lambda_{N_1}^1 = \lambda_n^0 + 2 \leq 2k + 2$. And we have $2^t k + 2 \geq 2^{t-1}k + 2 + 2$ $(n \geq 1)$ for $k \geq 2$. This completes the proof. □

In Fig.4, we plot the the distribution curve of $\lambda_{N^t}^t$ of G_t in semilog coordinate for the case of the initial graph G_0 is k-regular graph with $k = 3, 9, 27$.

4.3 Electrical network and resistance distance

For a graph G , we can define an electrical network [50], which is obtained from G by replacing each edge in G with a unit resistor. In the case without confusion, we also use G to express the electrical network corresponding to graph G . An important invariant of an electrical network is resistance distance [51]. For a pair of vertices i and j , their resistance distance $R(i, j)$ is defined to be the potential difference between i and j when a unit current is injected from i to j , which can be computed according to Ohms law. The sum of resistance distances between all pairs of vertices in G is called its Kirchhoff index [50], denoted by $R(G)$, namely,

$$R(G) = \sum_{1 \leq i < j \leq N} R(i, j). \quad (3)$$

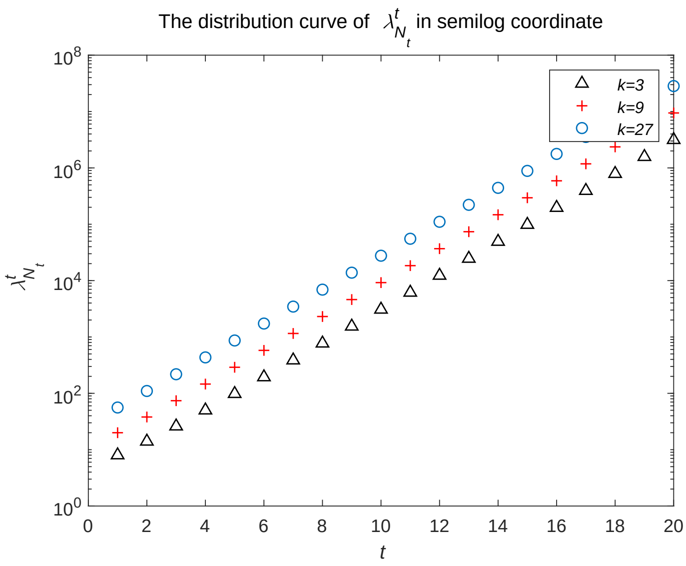

Figure 4: The distribution curve of $\lambda^t_{N^t}$ of G_t in semilog coordinate for the case of G_0 is k-regular graph with k=3, 9, 27.

It has been shown that [52] the Kirchhoff index of G can be determined in terms of the eigenvalues of $L(G)$.

For a connected graph G with order $N \geq 2$,

$$R(G) = N \sum_{i=2}^{N} \frac{1}{\lambda_i},$$

where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$ are the eigenvalues of matrix $L(G)$.

The resistance distance and Kirchhoff index have received considerable attention from the scientific community [53], since they have extremely useful connection with various fields, for example, random walks.

Let $R(G_t)$ denote the Kirchhoff index of the iterated heterologous bi-subdivision graphs G_t. This is

$$R(G_t) = N_t \sum_{i=2}^{N_t} \frac{1}{\lambda^t_i}.$$

According to Eq.(1) and Corollary 3.5.1, we can obtain N_t and the eigenvalues of Laplacian. Through computer numerical simulation, we can calculate $R(G_t)$ of the iterated heterologous bi-subdivision graphs.

In Fig.5, we plot the distribution curve of $R(G_t)$ in semilog coordinate for the case of the initial graph G_0 is 3-regular graph with 4 nodes and 6 edges.

The Kirchhoff index not only describes the size of the resistance network, but also reflects the connectivity of the network. According to Fig.5, we can see the distribution curve of Kirchhoff index of iterated heterologous bi-subdivision graphs. In the field of chemistry, Kirchhoff index can also be used to describe the structural properties of molecules and define the topological radius of polymers.

4.4 Consensus analysis

Distributed consensus is a fundamental problem in the context of multi-agent systems and distributed formation control. The consensus dynamics of multi-agent systems has gained much interest. Patterson and Bamieh proposed two norms quantifying as network coherence characterized by Laplacian spectrum to measure the deviation from consensus [54].

Firstly, the consensus dynamics of the following system is considered

$$\dot{x}(t) = -Lx(t) + \omega,$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_N(t))^\top$ denotes the state and subjects of the stochastic disturbances, ω is an N-vector of zero-mean white noise processes, and L is the Laplacian matrix.

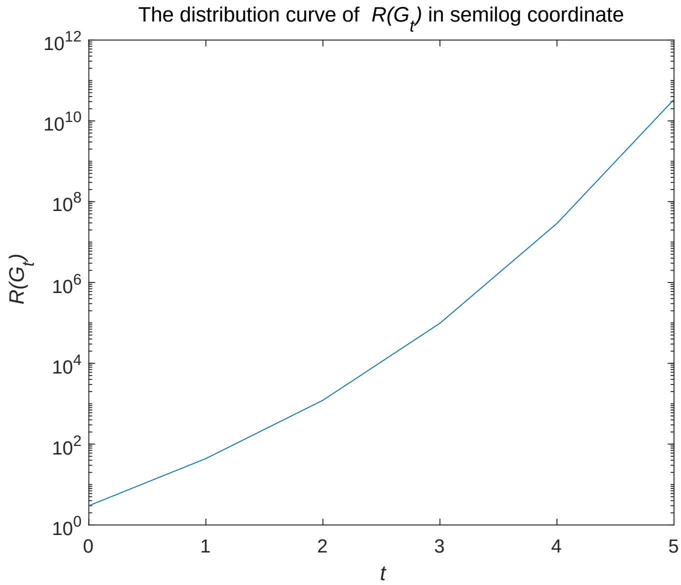

Figure 5: The distribution curve of $R(G_t)$ in semilog coordinate for the case of the initial graph G_0 is 3-regular graph with 4 nodes and 6 edges.

Then, the first-order network coherence is defined as the mean steady state variance of the deviation from the average of all node values, i.e.

$$H_{FO} := \lim_{t\to\infty} \frac{1}{N}\sum_{i=1}^{N} Var\Big(x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(t)\Big).$$

Let $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$ be the eigenvalues of Laplacian matrix. We can find that H_{FO} is fully determined by the Laplacian spectrum of the relationship of [55, 56], the first-order network coherence and Laplacian eigenvalues is as follows:

$$H_{FO} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i}.$$

And, the second-order network coherence is defined as the mean steady state variance of the deviation from the average of all vehicle positions. The equation represent as

$$H_{SO} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\lambda_i^2}.$$

Let H_{FO}^t and H_{SO}^t denote the first- and second-order network coherence of the iterated heterologous bi-subdivision graphs G_t. That is

$$H_{FO}^t = \frac{1}{2N_t}\sum_{i=2}^{N_t}\frac{1}{\lambda_i^t},$$

$$H_{SO}^t = \frac{1}{2N_t}\sum_{i=2}^{N_t}\frac{1}{(\lambda_i^t)^2}.$$

According to Eq.(1) and Corollary 3.5.1, we can obtain N_t and the eigenvalues of Laplacian. Through computer numerical simulation, we can calculate H_{FO}^t and H_{SO}^t of the iterated heterologous bi-subdivision graphs.

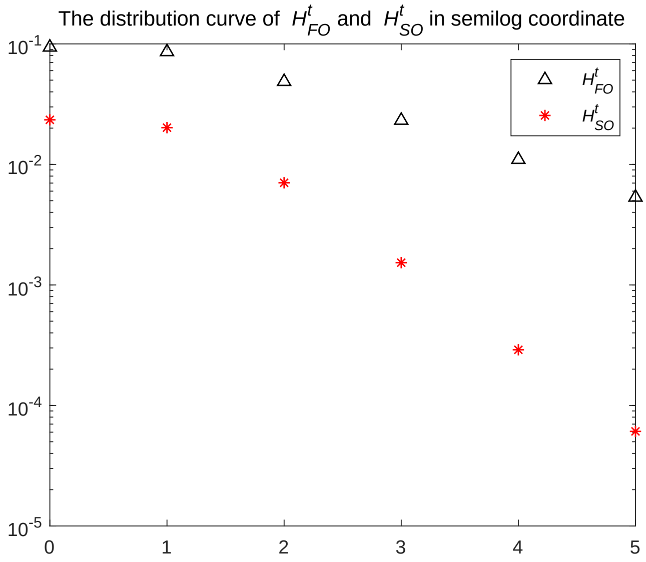

Figure 6: The distribution curve of H^t_{FO} and H^t_{SO} in semilog coordinate for the case of the initial graph G_0 is 3-regular graph with 4 nodes and 6 edges.

In Fig.6, we plot the distribution curve of H^t_{FO} and H^t_{SO} in semilog coordinate for the case of the initial graph G_0 is 3-regular graph with 4 nodes and 6 edges.

In this subsection, we obtain the scalings of the first and second-order network coherence. The obtained results show that n is bigger, $H^{(n)}_{SO}$ and $H^{(n)}_{FO}$ is smaller. It was surprising that the stability of the graph becomes more and more stable as the network scale becomes larger.

5 Conclusions

In summary, we first introduced heterologous bi-subdivision graph operation. According to this graph operation, we constructed iterated heterologous bi-subdivision graphs. Then, we studied the iterated formula of Laplacian eigenvalues of those model. Then, we gave some applications of Laplacian eigenvalues of iterated models, for example second smallest and largest eigenvalues, resistance distance, first- and second-order network coherence. Through the analysis of these variables, we find that the iterated heterologous bi-subdivision graphs have good topological properties and strong robustness.

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References

[1] D. Spielman, Spectral graph theory and its applications, in Proc. 48th Annu. IEEE Symp. Found. Comput. Sci., 29-38 (2007).

[2] L. Tran, V. Vu, and K. Wang, Sparse random graphs: Eigenvalues and eigenvectors, Random Struct. Algorithms, 42(1):110-134 (2013).

[3] J. Peng, E. Agliari, Z. Zhang, Exact calculations of first-passage properties on the pseudofractal scale-free web, Chaos: An Interdisciplinary Journal of Nonlinear Science, 25(7):073-118 (2015).

[4] D. Cvetković and S. Simić, Graph spectra in computer science, Linear Algebra Appl., 434(6):1545-1562 (2011).
[5] V. Mieghem, Graph Spectra for Complex Networks, Cambridge, U.K.: Cambridge Univ. Press (2011).
[6] J. Peng , G. Xu, Efficiency analysis of diffusion on T-fractals in the sense of random walks, The Journal of Chemical Physics, 140(13):134102 (2014).
[7] V. Mieghem and A. Jadbabaie, Moment-based spectral analysis of large-scale networks using local structural information, IEEE/ACM Trans. Netw., 21(2): 373-382 (2013).
[8] V. Mieghem, A. Jadbabaie, and G. Verghese, Structural analysis of Laplacian spectral properties of large-scale networks, IEEE Trans. Autom. Control, 58(9):2338-2343 (2013).
[9] M. Dai, T. Ju, Y. Zong, et al, The trapping problem of the weighted scale-free triangulation networks for biased walks, Fractals 27(3):1950028 (2018).
[10] M. Dai, W. Feng, X. Wu, et al, The Laplacian spectrum and average trapping time for weighted Dyson hierarchical network, Physica A: Statistical Mechanics and its Applications, 515:510-518 (2019).
[11] M. Dai, Y. Hou, C. Dai, et al, Characteristic polynomial of adjacency or Laplacian matrix for weighted treelike networks, Fractals, 27(5):1950074 (2019).
[12] L. Xi, Q. Ye, J. Yao, et al, Eigentime identities of flower networks with multiple branches, Physica A-statistical mechanics and its applications, 42 (2019).
[13] L. Xi, Q. Ye, Eigentime identities of potting networks, Physica A-statistical mechanics and its applications, 526 (2019).
[14] M. Niu, B. Xie, Regularity of a fractional partial differential equation driven by space-time white noise, Proc. Amer. Math. Soc., 138(4):1479-1489 (2010).
[15] Y. Zong, M. Dai, X. Wang, et al, Network coherence and eigentime identity on a family of weighted fractal networks, Chaos, Solitons and Fractals, 109: 184-194 (2018).
[16] Y. Chen, M. Dai, X. Wang, et al, Spectral analysis for weighted iterated triangulations of graphs, Fractals, 26(1):1850017 (2018).
[17] M. Hong, W. Sun, S. Liu, T. Xuan, Coherence analysus and Laplacian energy of recursive trees with controlled initial states, Frontiers of information Technology and Electronic Engineering (2019).
[18] X. Zhang, Y. Zhou, Q. Zhang, et al, Multi-objective Optimization of Barrier Coverage with Wireless Sensors, Evolutionary Multi-Criterion Optimization, Springer International Publishing, (2015).
[19] S. Cao and H. Qiu, Boundary Value Problems for harmonic functions on domains in Sierpinski gaskets, Constructive Approximation 2017(46):319-347 (2017).
[20] H. Qiu, R. S. Strichartz, Mean Value Properties of Harmonic Functions on Sierpinski Gasket Type Fractals. Journal of Fourier Analysis and Applications, 19(5):943-966 (2013).
[21] S. Cao and H. Qiu, Some Properties of the Derivatives on Sierpinski Gasket Type Fractals, Constr. Approx 46:319-347 (2017).
[22] Hasunuma, Toru, Structural properties of subdivided-line graphs, Journal of Discrete Algorithms, 31:69-86 (2015).
[23] Y. Qi, Z. Zhang, Spectral properties of extended Sierpiński graphs and their applications. IEEE Transactions on Network Science and Engineering, 1-1 (2018).
[24] F. R. K. Chung, *Spectral graph theory*, American Mathematical Society (2012).
[25] Y. Zeng, Z. Zhang, Spectra, hitting times, and resistance distances of q-subdivision graphs, Theoretical Computer Science (2018).
[26] B. Wu, Z. Zhang, W. Su, Spectral analysis for weighted iterated q-triangulation networks, Chaos (2019).
[27] A. Aytaç, On the edge-tenacity of the middle graph of a graph, International Journal of Computer Mathematics, 82(5):551-558 (2005).
[28] Y. Qi, Z. Zhang, Spectral properties of extended Sierpiński graphs and their applications, IEEE Transactions on Network Science and Engineering, 1-1 (2018).
[29] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs, Theory and Applications*, Third edition, Johann Ambrosius Barth. Heidelberg, (1995).
[30] R. Bapat, $Graphsandmatrices$, Springer, (2010).
[31] H. Sachs, Über Teiler, Faktoren und charakteristische Polynome von Graphen, $Teil\ II$, Wiss. Z. TH Ilmenau 13:405-412 (1967).
[32] M. Fiedler, Algebraic Connectivity of Graphs, Czechoslovak Math. J. 23:298-305 (1973).
[33] F.R.K. Chung, Spectral graph Theory, CBMS, American Mathematics Society, (1997).
[34] C. Godsil, G. Royle, Algebraic Graph Theory, GTM, Springer, (2001).

[35] R. Grone, R. Merris, Algebraic connectivity of trees, Czechoslovak Math. J. 37:660-670 (1987).
[36] S. Kirkland, M. Neumann, Algebraic connectivity of weighted trees under perturbations, Linear Algebra Appl. 40:187-203 (1996).
[37] R. Merris, Characteristic vertices of trees, Linear and Multilinear Algebra 22:115-131 (1987).
[38] A. Berman, X. Zhang, Lower bounds for the eigenvalues of Laplacian matrices, Linear Algebra Appl. 316:13-20 (2000).
[39] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Math. J. 25:607-618 (1975).
[40] D. Zuckerman, A technique for lower bounding the cover time, SIAM J. Discrete Math., 5(1):81-87 (1992).
[41] S. Ikeda, I. Kubo, and M. Yamashita, The hitting and cover times of random walks on finite graphs using local degree information, Theoret. Comput. Sci., 410(1):94-100 (2009).
[42] A. Broder and A. Karlin, Bounds on the cover time, J. Theor. Probab., 2(1):101-120 (1989).
[43] B. Mohar, The Laplacian spectrum of graphs, Graph Theory Combin. Appl. 2:871-898 (1991).
[44] F. Chung, V. Faber, T. Manteuffel, An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian, SIAM J. Discrete Math. 7(3):443-457 (1994).
[45] B. Mohar, Eigenvalues, diameter, and mean distance in graphs, Graphs Combin. 7:53-64 (1994).
[46] I. Gutman, V. Gineityte, M. Lepovic, M. Petrovi, The high-energy band in the photoelectron spectrum of alkanes and its dependence on molecular structure, J. Serb. Chem. Soc. 64:673-680 (1999).
[47] C. Delorme, S. Poljak, Laplacian eigenvalues and the maximum cut problem, Math. Program. 62:557-574 (1993).
[48] P. Sole, Expanding and forwarding, Discrete Appl. Math. 58:67-78 (1995).
[49] J. Li, W. Shi, A. Chang, The Laplacian spectral radius of graphs, Czechoslovak Mathematical Journal, 60(3):835-847 (2010).
[50] P. Doyle and J. Snell, Random Walks and Electric Networks. Mathematical Association of America, (1984).
[51] D. Klein and M. Randic, Resistance distance, J. Math. Chem., 12(1):81-95 (1993).
[52] I. Gutman and B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, J. Chem. Inf. Comput. Sci., 36(5):982-985 (1996).
[53] A. Ghosh, S. Boyd, and A. Saberi, Minimizing effective resistance of a graph, SIAM Rev., 50(1):37-66 (2008).
[54] S. Patterson and B. Bamieh, Network Coherence in Fractal Graphs, IEEE Conf. Decision and Control and European Control Conf. 413(1):6-44 (2011)
[55] G. Young, L. Scardovi and N. E. Leonard, Robustness of noisy consensus dynamics with directed communication, Proceedings of the 2010 American Control Conf, 6312-6317 (2010).
[56] L. Xiao, S. Boyd and S. J. Kim, Distributed average consensus with least-mean-square deviation, J. Parallel Distrib. Comp. 67(1):33-46 (2007).