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Existence of Nontrivial Solutions for Schrödinger-Possion System with Sign-Changing Potentials

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Abstract: In the present paper, we study the existence of nontrivial solutions of a class of Schrödinger-Poisson system. Comparing the previous works, the novelty is that we encounter some new challenges for the general nonlinearities with non-local terms. By doing some delicate estimates for the nonlocal term, we overcome the difficulty and prove the existence of nontrivial solution for the system.

Keywords: Nonlinear Schrödinger-Poisson systems; Nontrivial solutions; Variational methods.

1 Introduction and main results

In this paper, we consider the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u = K(x)\phi(x)u + f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where V is non-periodic in x and changes sign, f is an asymptotically linear function at infinity. From physical point of view, the system (1) also arises in many fields. For instance, one considers the following system

$$\begin{cases} i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + k(x)\psi + K(x)\phi\psi - h(x)m(|\psi|)\psi, & x \in \mathbb{R}^3,\ t \in \mathbb{R}, \\ -\frac{\hbar^2}{2m}\Delta\phi(x) = K(x)|\psi|^2,\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{2}$$

where $\hbar$ is the Planck constant, i is the imaginary unit and Δ is the Laplacian operator. A standing wave is a solution of (2) of the form $\psi(x,t) = u(x)e^{i\omega t}$, $\omega > 0$ and $t \in \mathbb{R}$. It is quite clear that $\psi(x,t)$ solves (2) if and only if $u(x)$ solves the so-called stationary equations

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + K(x)\phi(x)u = h(x)f(u), & x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{3}$$

where $\varepsilon = \sqrt{\frac{\hbar^2}{2m}}$, $V(x) = k(x) + \omega\hbar$ and $f(u) = m(|u|)u$. The system (2) simulates some physical phenomena, for example, in quantum electrodynamics, semiconductor theory, nonlinear optics and plasma physics, see [6, 7, 29] and in Abelian Gauge Theories, see [11, 12, 23], and the references therein. According to the classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling nonlinear Schrödinger-Poisson and Maxwell equations. Benci and Fortunato [7] studied electrostatic situations where there is the interaction between electrostatic field and solitary wave in nonlinear steady equation. At present we consider the case when $\hbar$ is constant. Without loss of generality we can assume that $\varepsilon = 1$. For this case many papers deal with the existence of positive ground and bound state solutions, for example, one can refer to [2, 3, 7, 13, 17, 18, 24, 25, 34, 37] and references therein. We point out that the difficulty we have to face, which are related to the potential and the unboundedness of the space $\mathbb{R}^3$ when dealing with (1)

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or (3). About the autonomous case and the case in which the coefficients are supposed to radial, have been investigated in the last decades by many authors. (see [3, 25, 26]), just to overcome the lack of compactness-taking advantage of the compact embedding of the subspace of $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)(\forall p \in (2, 6))$. More recently, many of the contributions of (3) also give a case where the coefficients in (1) do not give a symmetry assumption. (see [1, 8, 9, 15]).

Up till now, a large number of the works has been done in the past, for instance, Zhao in [36] considered the system (3) when $\varepsilon = 1$, $f(u) = |u|^{p-2}u(2 < p < 6)$, studying the existence and concentration of solutions for the Schrödinger-Poisson equations

$$\begin{cases} -\Delta u + V(x)u + \lambda K(x)\phi(x)u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & x \in \mathbb{R}^3, \end{cases}$$

with steep well potential. Given the similar assumptions about V and K, f is either superlinear or sublinear in u as $|u| \to \infty$. For some more generic 4-superlinear conditions on $f(x, u)$, Ye and Tang in [35] have proved the existence and multiplicity of solutions for the autonomous case

$$\begin{cases} -\Delta u + V(x)u + \lambda K(x)\phi(x)u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & x \in \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter. This paper devoted to studying the existence of a solution of the following Schrödinger-Poisson systems where f is an asymptotically linear nonlinearity, and K, V such that $\lim_{|x|\to\infty} K(x) = 0$ and $\lim_{|x|\to\infty} V(x) = V_\infty$.

In this paper, the motivation to looking for a nontrivial solution to Equation (1) is inspired by the research of existence of a nontrivial solution for nonlinear Schrödinger equations

$$-\triangle u + V(x) = f(u) \quad \text{in } H^1(\mathbb{R}^N) \tag{4}$$

in $\mathbb{R}^N$ (see[20]). In recent years, this system has been widely studied under variant assumptions on V and f. Their proof is based on a decomposition of $H^1(\mathbb{R}^3)$ into two infinite-dimensional subspaces and a generalised linking theorem, that is applied to the energy functional related to the problem. Especially in [21], Maia and Soares found a radial nontrivial solution for the radial nonlinear Schrödinger equation when $N \geq 3$. To dealing with the lack of compactness of the Sobolev embeddings in $\mathbb{R}^N$, it is easy to use the Concentration-compactness Principle of Lions to compensate for compactness. In order to obtain the linking geometry in Cerami condition as in [20], it is crucial to projected the limit problem

$$-\triangle u + V_\infty u = f(u) \quad \text{in } H^1(\mathbb{R}^3) \tag{5}$$

on a infinite-dimensional subspace of $H^1(\mathbb{R}^3)$ with finite codimension (see[22, Theorem 1] or [27, Theorem 1]). Here w_0 is a unique positive ground-state solution of the limit problem, (proved in [5]).

In order to prove our main results, we assume that:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $-V_0 \leq V(x) < V_\infty$, where $V_0, V_\infty > 0$.

(V_2) $\lim_{|x|\to\infty} V(x) = V_\infty$.

(V_3) $0 \in \sigma(L)$ and $\inf \sigma(L) < 0$, where $\sigma(L)$ is the spectrum of the operator $L = -\triangle + V$.

(V_4) $K(x) \in L^2(\mathbb{R}^3)$ is a nonnegative function, $K(x) > 0$, $\lim_{|x|\to\infty} K(x) = 0$.

(V_5) $V(x) \leq V_\infty - C_\zeta e^{-\zeta|x|}$, where $C_1 > 0$ and $0 < \zeta < \sqrt{V_\infty}$.

We set

$$F(u) := \int_0^u f(l)dl,$$

and make the following hypotheses on f.

(f_1) $f \in C^3(\mathbb{R}, \mathbb{R})$ and $\lim_{l\to 0}(f(l)/l) = 0$.

(f_2) $\lim_{|l|\to\infty}(|f(l)|/|l|) = m > V_\infty$ and $|f(l)|/|l| < m$ for all $l \in \mathbb{R}\backslash\{0\}$.

(f_3) $Q(l) = f(l)l - 2F(l)$; then, for all $l \in \mathbb{R}\backslash\{0\}$, $F(l) \geq 0,\ Q(l) > 0$,

$$\lim_{l\to\infty} Q(l) = +\infty.$$

(f_4) There are $C_k > 0$ and $3 < p_1 \leq p_2$, such that $p_1, p_2 < 5$ and

$$|f^{(k)}(l)| \leq C_k(|l|^{p_1-k} + |l|^{p_2-k}), \text{ for } k \in \{0,1,2,3\} \text{ and } l \in \mathbb{R}.$$

(f_5) The function $l \mapsto f(l)/l$ is increasing in $l \in (0, +\infty)$.

This kind of hypotheses was first introduced by Liliane in [20], We used it in expressing our results.

Theorem 1 *Assume that* (V_1)-(V_5) *and* (f_1)-(f_5) *hold. Then problem* (1) *has a nontrivial weak solution* u *in* $H^1(\mathbb{R}^3)$.

The paper is organized as follows. The proof of theorem 1 contain in section 4. In Section 2, we devoted to the construction of the variational framework, to recalling some known results and to stating some useful estimates. Section 3 is dedicated to show the functional I satisfies the geometry of the linking theorem, and we prove our main result to get the boundedness of the Cerami sequences of the functional I associated with problem (1). In Section 4, we prove that the boundedness of the Cerami sequences of the functional I associated with problem (1).

Remark 2 *It is easy to verify that the asymptotically cubic nonlinearity* $f(u) = \frac{u^3}{1+nu^2}$ *with* $n \in (0, V_\infty^{-1})$ *satisfy hypotheses* (f_1)-(f_5).

2 Variational setting

Throughout the paper, we shall introduce the notation and terminology in expressing our results. We define the operator $\Phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ as

$$\Phi(u) = \phi_u = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy = \frac{1}{|x|} * K(x)u^2.$$

From [9], [8], we know that ϕ_u is a weak solution of $-\Delta\phi = K(x)u^2$. If $u_n \rightharpoonup u \in H^1(\mathbb{R}^3)$ then $\phi(u_n) \to \phi(u)$ in $D^{1,2}(\mathbb{R}^3)$. Moreover, $\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 \to \int_{\mathbb{R}^3} K(x)\phi_u u^2$ and $\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n\phi \to \int_{\mathbb{R}^3} K(x)\phi_u u\phi, \forall\phi \in H^1(\mathbb{R}^3)$.

Let $E := H^1(\mathbb{R}^3)$. The energy functional $J_V : E \to \mathbb{R}$ with equation (1) is defined by

$$J_V(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \tag{1}$$

with $u \in E$. From conditions (V_2) and (V_3), it is straightforward to show that there is a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k < 0$ from the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u,\ \ u \in L^2(\mathbb{R}^3). \tag{2}$$

(see [14, Theorem 30],[20]). Indicated by φ_i ,the eigenfunction corresponding to λ_i, $i \in 1, 2, ..., k$, in $H^1(\mathbb{R}^3)$. Setting

$$E^- := \operatorname{span}\{\varphi_i : i = 1, 2, \ldots, k\} \text{ and } E^+ := \left(E^-\right)^\perp,$$

we can get $E = E^+ \oplus E^-$. Refering to [28, Theorem 3.15], the essential spectrum of $-\triangle + v$ is the interval $[V_\infty, +\infty)$, that is, dim $E^- < \infty$. Thereafter, for every function $u \in E$ can be written as $u = u^+ + u^-$ uniquely, where $u^+ \in E^+$ and $u^- \in E^-$. Therefore, by following the same method in [20], we may introduce the new inner product $\langle\cdot,\cdot\rangle$ in E, namely

$$\langle u, v\rangle = \begin{cases} \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + V(x)uv)\mathrm{d}x & \text{if } u, v \in E^+, \\ -\int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + V(x)uv)\mathrm{d}x & \text{if } u, v \in E^-, \\ 0 & \text{if } u \in E^+ \text{and } v \in E^-, \end{cases}$$

such that the corresponding norm $\|\cdot\|$ is equivalent to $\|\cdot\|_E$, where $\|\cdot\|_E$ denotes a standard norm in $H^1(\mathbb{R}^3)$. Moreover, following from (1), the functional J_V may be written as

$$J_V(u) = \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \tag{3}$$

for every function $u = u^+ + u^- \in E$. Owing to the definition of φ_i and $\lambda_i \in 1, 2, ..., k$, notice that

$$\int_{\mathbb{R}^3} u^+(x)u^-(x)\mathrm{d}x = 0 \tag{4}$$

for all functions $u^+ \in E^+$ and $u^- \in E^-$.

Given $(X, \|\cdot\|)$ is a Banach space, for some constant $M > 0$, we can say that $\{z_n\} \subset X$ is Cerami sequence if $|J(z_n)| < M$. What's more, a functional $J \in C^1(X, \mathbb{R})$ satisfies the Cerami condition (C) refer to $\|J'(z_n)\|_{E^*}(1 + \|z_n\|) \to 0$ has a subsequence $z_{n_k} \to z$ in X.

3 A nontrivial critical point

Theorem 3 ***(Linking Theorem under the*** (C_c) ***condition).*** *Let $R > \rho > 0$, $E = E^+ \bigoplus E^-$ be a Banach space with dim $E^- < \infty$. And let $w \in E^+$ be a fixed element such that $||u|| = \rho$. Setting :*

$$M := \{z = su + v^- : \|z\| \le R,\ s \ge 0,\ v^- \in E^-\},$$
$$M_0 := \{z = su + v^- : v^- \in E^-,\ \|z\| = R,\ s \ge 0 \textit{ or } \|z\| \le R,\ s = 0\},$$
$$N_\rho := \{z \in E^+ : \|z\| = \rho\}.$$

Let $I \in C^1(E, \mathbb{R})$ be such that

$$\alpha_1 := \inf_{N_\rho} J_V > \beta := \max_{M_0} J_V.$$

Then, $c \ge \alpha$, and there exists a Cerami sequence at level c for the functional J_V where

$$c := \inf_{\gamma \in \Gamma} \max_{z \in M} J_V(\gamma(z)), \quad \gamma := \left\{\gamma \in C(M, E) : \gamma|_{M_0} = Id\right\}.$$

To simplify the notation, given $z \in E$ and $y \in \mathbb{R}^3$, we write $z^+(\cdot - y)$ (or $z^-(\cdot - y)$) referring to the projection in E^+ (respectively, in E^-) of the translated function $z(\cdot - y)$. Consider the limit problem (5) and the energy functional associated with the equation (5) given by

$$J_\infty(z) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla z|^2 + V_\infty z^2\right)\mathrm{d}x - \int_{\mathbb{R}^3} F(z)\mathrm{d}x,\ z \in H^1(\mathbb{R}^3),$$

for $z \in E$. Let $w_0 \in E$ be a positive, and continuous solution of the equation (5) such that $J_\infty(w_0) = c_\infty > 0$. Due to $V_\infty < m$, it is not difficult to directly check the existence of w_0.

In order to prove that J satisfies linking theorem, we need to state some basic conclusions.

Remark 4 *If z and v are functions in $L^2(\mathbb{R}^3)$, it holds that*

$$\int_{\mathbb{R}^3} z(x - y)v(x)\mathrm{d}x \to 0 \quad \textit{if } |y| \to \infty.$$

For $R > 0$ and $y \in \mathbb{R}^3$, consider

$$M = \{z = sw_0^+(\cdot - y) + v^- : v^- \in E^-,\ ||z|| \le R,\ s \ge 0\}$$

and

$$M_0 = \{z = sw_0^+(\cdot - y) + v^- : v^- \in E^-,\ ||z|| = R,\ s \ge 0 \textit{ or } ||z|| \le R, s = 0\}.$$

Remark 5 *There exist constants $\alpha > 0$,such that $N_\rho \subset E^+$ and $J_V|_{N_\rho} \ge \alpha$.*

Proof. On account of assumptions (f_1) and (f_2), given $\varepsilon > 0$ and $2 \le p \le 2^*$, there exists $C, C_\varepsilon > 0$ such that

$$|F(l)| \le \varepsilon|l|^2 + C_\varepsilon|l|^p \text{ and } |f(l)| \le \varepsilon|l| + C_\varepsilon|l|^{p-1}, \tag{5}$$

for all $l \in \mathbb{R}$.
Since $N_\rho \subset E^+$, $2 \leq p \leq 2^*$, by (5) for all $z \in N_\rho$, thanks to the equivalence of the norms, it yields

$$\begin{aligned} J_V(z) &= \frac{1}{2}\|z\| - \int_{\mathbb{R}^3} F(x, z(x))dx - \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_z z^2 dx \\ &\geq \frac{1}{2}\rho^2 - \int_{\mathbb{R}^3} (\varepsilon|z(x)|^2 + C_\varepsilon|z(x)|^p)dx - \frac{C}{4}\|z\|^4 \\ &\geq \frac{1}{2}\rho^2 - (\varepsilon C_2^2\|z\|^2 + C_\varepsilon C_p^p\|z\|^p + \frac{C}{4}\|z\|^4) \\ &= \rho^2\left[(\frac{1}{2} - \varepsilon C_2^2) - (C_\varepsilon C_p^p \rho^{p-2} - \frac{C}{4}\rho^2)\right] \\ &\geq \rho^2(b_1 - b_2) = \alpha > 0, \end{aligned} \tag{6}$$

We choose ε, ρ are sufficiently small, such that $\frac{1}{2} > \varepsilon C_2^2$, $\frac{C_\varepsilon}{p}C_p^p\rho^{p-2} > \frac{C}{4}\rho^2$ and also

$$b_1 := (\frac{1}{2} - \varepsilon C_2^2) > C_\varepsilon C_p^p \rho^{p-2} - \frac{C}{4}\rho^2 =: b_2,$$

$\alpha > 0$ is guaranteed and from (6), it is clear that $I|_{N_\rho} \geq \alpha$. ■

Remark 6 ([20]APPENDIX A.) *Denote B_1 is the open ball of radius 1 with the boundary ∂B_1, where the open ball in a finite-dimensional space generated by the functions $w_0^+, \varphi_1, ..., \varphi_n$, such that*

$$\lim_{R\to\infty} \int_{\mathbb{R}^3} \left(\frac{F(Ru)}{(Ru)^2} - \frac{m}{2}\right) u^2 \mathrm{d}x = 0, \tag{7}$$

uniformly for $u \in \partial B_1$.

Proof. In truth, for each $R = j \in \mathbb{N}$, consider $G_j : \partial B_1 \to \mathbb{R}$ the functional given by $G_j = \int_{\mathbb{R}^3} \left(\frac{F(ju)}{(u)^2} - \frac{m}{2}\right) u^2 \mathrm{d}x$. The continuity of the function F shows that G_j is a continuous functional for each fixed j. The equivalence of the norms and hypothesis (f_2) show that there exist a constant $P > 0$ such that

$$0 \leq G_j(u) = \int_{\mathbb{R}^3} \left(\frac{F(ju)}{(ju)^2} - \frac{m}{2}\right) u^2 \mathrm{d}x \leq P$$

for all ∂B_1. Because of the continuity of the functional G_j in the compact set ∂B_1, for each fixed j, the functional G_j assumes its maximum at $u_j \in \partial B_1$. Consider finite dimensional space spanned by the functions $w_0^+, \varphi_1, ..., \varphi_n$ and the maximum sequence u_j, since $||u|| = 1$ for each j, here exists $u^* \in \partial B_1$ such that, up to a subsequence,

$$u_j \to u^*, \tag{8}$$

strongly in the norm $|| \cdot ||$. We have $0 \leq G_j(u) \leq G_j(u_j)$ for all $u_j \in \partial B_1$ and for each j, thus,

$$0 \leq \int_{\mathbb{R}^3} \left(\frac{F(ju)}{(ju)^2} - \frac{m}{2}\right) u^2 \mathrm{d}x \leq \int_{\mathbb{R}^3} \left(\frac{F(ju_j)}{(ju_j)^2} - \frac{m}{2}\right) u_j^2 \mathrm{d}x \tag{9}$$

for all u and for each j. Taking the limit $j \to \infty$, we first note that $u_j \to u^*$ almost everywhere in $\mathbb{R}^3$. Thus, if $u^* \neq 0$, it follows that $ju_j(x) \to \infty$ if $j \to \infty$. Hence, hypothesis (f_2) yields

$$\left(\frac{F(ju_j(x))}{(ju_j(x))^2} - \frac{m}{2}\right) u_j(x)^2 \to 0 \tag{10}$$

if $j \to \infty$. If $u^*(x) = 0$, we also have (10). By the strong convergence in (8), there exists a function $g \in L^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$0 \leq \left(\frac{F(ju_j(x))}{(ju_j(x))^2} - \frac{m}{2}\right) u_j(x)^2 \leq m|u_j^2(x)| \leq mg(x) \in L^1(\mathbb{R}^3). \tag{11}$$

Finally, by (10) and (11), the Lebesgue Dominated Convergence Theorem guarantee

$$\lim_{j\to\infty}\int_{\mathbb{R}^3}\left(\frac{F(ju_j)}{(ju_j)^2}-\frac{m}{2}\right)u_j^2\mathrm{d}x=0. \tag{12}$$

It is clear that, taking $j\to\infty$ in (9), we have

$$\lim_{j\to\infty}\int_{\mathbb{R}^3}\left(\frac{F(ju)}{(ju)^2}-\frac{m}{2}\right)u^2\mathrm{d}x=0, \tag{13}$$

uniformly for $u\in\partial B_1$. ■

Next we shall prove the compactness lemma for the functional J_V.

Lemma 7 *There exist $R>0$ and $y\in\mathbb{R}^3$, with R and $|\ y\ |$sufficiently large, such that $J_V|_{M_0}\leq 0$.*

Proof. Since subset M_0 is equal to a disjioint union of M_1 and M_2 denote

$$M_1\ =\{z=sw_0^+(\cdot-y)+v^-:v^-\in E^-,\ ||z||\leq R,\ s=0\}$$

and

$$M_2\ =\{z=sw_0^+(\cdot-y)+v^-:v^-\in E^-,\ ||z||=R,\ s>0\}.$$

Since $M_1\subset E^-$, we have $J_V(z)<0$ for any $z\in M_1$. Let $R>0$ and $z\in M_2$ with $\|z\|=R$. Writing

$$z=||z||z/||z||=||z||u(z)=||z||(\lambda(z)w_0^+(\cdot-y)+v^-)$$

where $\lambda\in\mathbb{R}^+, v^-\in\mathbb{E}^-$ and $||u(z)||=1$, this is,

$$\begin{aligned}J_V(z)&=J_V(u(z)||z||)\\&=\frac{1}{2}||z||^2\lambda^2(z)||w_0^+(\cdot-y)||^2-\frac{1}{2}||z||^2||v^-||^2-\frac{1}{4}\int_{\mathbb{R}^3}\phi_{||z||u(z)}||z||^2u^2(z)\mathrm{d}x-\int_{\mathbb{R}^3}F(||z||u(z))\mathrm{d}x\\&=\frac{1}{2}||z||^2\left\{\lambda^2(z)||w_0^+(\cdot-y)||^2-\frac{1}{2}||v^-||^2-\frac{1}{4}\int_{\mathbb{R}^3}\phi_{||z||u(z)}u^2(z)\mathrm{d}x\ -2\int_{\mathbb{R}^3}\frac{F(Ru(z))}{(Ru(z))^2}u^2(z)\mathrm{d}x\right\}.\end{aligned} \tag{14}$$

For convenience we simplify the notation, write $\lambda,u,$ and v^- as $\lambda(z)$, $u(z)$ and $v^-(z)$, respectively. By hypothesis (f_2), $lim_{|t|\to\infty}(|F(l)|/|l|^2)=\frac{1}{2}m$ and $|F(l)|/|l|^2\leq\frac{m}{2}$ for all $l\neq0$, we then have

$$\left\|\frac{F(Ru)}{(Ru)^2}u^2\right\|\leq\frac{m}{2}u^2\in L^1(\mathbb{R}^3).$$

In light of Remark 6,

$$\lim_{R\to\infty}\int_{\mathbb{R}^3}\left(\frac{F(Ru)}{(Ru)^2}-\frac{m}{2}\right)u^2\mathrm{d}x=0 \tag{15}$$

for all $u\in E$ such that $\|u\|=1$. Since M_2 is contained in a finite-dimensional subspace of $E, z=\|z\|u(z)\in M_2$, with $\|u\|=1$, having made these considerations, the limit (15) is then uniform in u. In addition, from $\int_{\mathbb{R}^3}w_0^+(x-y)v^-\mathrm{d}x=0$, we can get

$$\begin{aligned}J_V(z)&=\frac{1}{2}||w||^2\left[\lambda^2||w_0^+(\cdot-y)||^2-\frac{1}{2}||v^-||^2-\frac{1}{4}\int_{\mathbb{R}^3}\phi_{||z||u}u^2\mathrm{d}x-m\int_{\mathbb{R}^3}u^2(z)\mathrm{d}x+o_R(1)\right]\\&=\frac{1}{2}||z||^2\left[\lambda^2||w_0^+(\cdot-y)||^2-\frac{1}{2}||v^-||^2-\frac{1}{4}\int_{\mathbb{R}^3}\phi_{||z||u}u^2\mathrm{d}x-m\int_{\mathbb{R}^3}(\lambda w_0^+(x-y)+v^-)^2\mathrm{d}x+o_R(1)\right]\\&=\frac{1}{2}||z||^2\left[\lambda^2||w_0^+(\cdot-y)||^2-||v^-||^2-\frac{1}{4}\int_{\mathbb{R}^3}\phi_{||z||u}u^2\mathrm{d}x-m\lambda^2\int_{\mathbb{R}^3}(w_0^+)^2(x-y)\mathrm{d}x\right.\\&\quad\left.-m\int_{\mathbb{R}^3}(v^-)^2\mathrm{d}x+o_R(1)\right]\\&\leq\frac{1}{2}||z||^2\left[\lambda^2\left(||w_0^+(\cdot-y)||^2-m\int_{\mathbb{R}^3}(w_0^+)^2(x-y)\mathrm{d}x\right)+o_R(1)\right].\end{aligned} \tag{16}$$

According to hypothesis $(V_1), V(x) \leq V_\infty$ for all $x \in \mathbb{R}^3$, indicating that

$$\begin{aligned} \left\|w_0^+(\cdot - y)\right\|^2 &= \int_{\mathbb{R}^3} \left(\left|\nabla w_0^+(x-y)\right|^2 + V(x)\left(w_0^+\right)^2(x-y)\right)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} \left(\left|\nabla w_0^+(x-y)\right|^2 + V_\infty\left(w_0^+\right)^2(x-y)\right)\mathrm{d}x \\ &= \left\|w_0^+(\cdot - y)\right\|_{V_\infty}^2 \leq \left\|w_0(\cdot - y)\right\|_{V_\infty}^2 . \end{aligned} \tag{17}$$

Because of the translational invariance of J_∞, then w_0 and $w_0(\cdot - y)$ are critical points of the functional of I_∞. Hence, $J'_\infty(w_0(\cdot - y))w_0(\cdot - y) = 0$, that is,

$$J'_\infty(w_0(\cdot - y)) = \langle Lw_0, w_0\rangle - \int_{\mathbb{R}^3} f(w_0(x-y))(w_0(x-y))\mathrm{d}x.$$

$$\|w_0(\cdot - y)\|_{V_\infty}^2 = \int_{\mathbb{R}^3} f(w_0(x-y))(w_0(x-y))\mathrm{d}x. \tag{18}$$

From (17) and (18),

$$\left\|w_0^+(\cdot - y)\right\| \leq \int_{\mathbb{R}^3} f(w_0(x-y))(w_0(x-y))\mathrm{d}x. \tag{19}$$

Subtracting (16) in (19), and then afterwards adding and subtracting the integral $\int_{\mathbb{R}^3} w_0^2(x-y)\mathrm{d}x$, we obtain

$$\begin{aligned} J_V(z) \leq& \frac{1}{2}\|z\|^2 \left\{\lambda^2 \left[\int_{\mathbb{R}^3} f\left(w_0(x-y)\right) w_0(x-y)\mathrm{d}x \right.\right. \\ &\left.\left. -m\int_{\mathbb{R}^3} \left(w_0^+\right)^2(x-y)\mathrm{d}x\right] + o_R(1)\right\} \\ =& \frac{1}{2}\|z\|^2 \left\{\lambda^2 \left[\int_{\mathbb{R}^3} f\left(w_0(x-y)\right) w_0(x-y)\mathrm{d}x \right.\right. \\ &\left.\left. -m\int_{\mathbb{R}^3} w_0^2(x-y) - \left(w_0^+\right)^2(x-y)\right]\mathrm{d}x\right] + o_R(1)\right\} \\ =& \frac{1}{2}\|z\|^2 \left\{\lambda^2 \left[\int_{\mathbb{R}^3} f\left(w_0(z)\right) w_0(z)\mathrm{d}z - m\int_{\mathbb{R}^3} w_0^2(z)\mathrm{d}z \right.\right. \\ &\left.\left. +m\int_{\mathbb{R}^3} \left[w_0^2(x-y) - \left(w_0^+\right)^2(x-y)\right]\mathrm{d}x\right] + o_R(1)\right\}. \end{aligned} \tag{20}$$

At this point, we figure out two integrals:

$$m\int_{\mathbb{R}^3} \left[w_0^2(x-y) - \left(w_0^+\right)^2(x-y)\right]\mathrm{d}x \tag{21}$$

and

$$\int_{\mathbb{R}^3} f\left(w_0(u)\right) w_0(u)\mathrm{d}u - m\int_{\mathbb{R}^3} w_0^2(u)\mathrm{d}u. \tag{22}$$

First we estimate (21). Because of $\int_{\mathbb{R}^3} w_0^+(x-y)w_0^-(x-y)\mathrm{d}x = 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} [w_0^2(x-y) - \left(w_0^+\right)^2(x-y)]\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^3} \{\left[w_0^+(x-y) + w_0^-(x-y)\right]^2 - \left(w_0^+\right)^2(x-y)\}\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^3} \left[\left(w_0^+\right)^2(x-y) + \left(w_0^-\right)^2(x-y) - \left(w_0^+\right)^2(x-y)\right]\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^3} \left(w_0^-\right)^2(x-y)\mathrm{d}x. \end{aligned}$$

Claim 8 *The integral $\int_{\mathbb{R}^3}\left(w_0^-\right)^2(x-y)\mathrm{d}x \to 0$ as $|y|\to\infty$. Indeed, given $\varepsilon>0$, since $\varphi_1,...,\varphi_n$ is a basis of eigenfunctions for the subspace E^-, Remark 4 and hypothesis (V_1) ensure that, for each $i\in\{1,...,n\}$, there exists $M_i>0$ such that if $|y|\geq M_i$, then*

$$\begin{aligned}&\langle w_0(x-y),\varphi_i\rangle\\&\quad=\tfrac{1}{2}\int_{\mathbb{R}^3}\nabla w_0(x-y)\cdot\nabla\varphi_i(x)\mathrm{d}x+\tfrac{1}{2}\int_{\mathbb{R}^3}V(x)w_0(x-y)\varphi_i(x)\mathrm{d}x<\varepsilon.\end{aligned}$$

Proof. Taking $\hat{M}=max\{M_1,...,M_n\}$, thus it can be seen

$$\langle w_0(x-y),\varphi_i\rangle<\varepsilon \quad for\ \ all\ \ i\in\{1,...,n\},\ if\ \ \ |y|\geq\hat{M}. \tag{23}$$

Since $w_0(\cdot-y)\in E^-$ is a linear combination of the vectors $\varphi_1,...,\varphi_n$, namely,

$$w_0^-(x-y)=\sum_{i=1}^n\xi_i(y)\varphi_i(x).$$

It follows from (20) that there exists $\hat{M}>0$ such that, if $|y|\geq\hat{M}$, then

$$\begin{aligned}\left\|w_0^-(\cdot-y)\right\|^2&=\left\langle w_0(\cdot-y),w_0^-(\cdot-y)\right\rangle\\&=\langle w_0(\cdot-y),\sum_{i=1}^n\xi_i(y)\varphi_i(\cdot)\rangle\\&<\varepsilon n\max\left\{\left|\xi_1(y)\right|,\ldots,\left|\xi_n(y)\right|\right\}.\end{aligned}\tag{24}$$

Claim 9 *There exists a constant $H>0$ that does not depend on y, such that*

$$\max\left\{\left|\xi_1(y)\right|,\ldots,\left|\xi_n(y)\right|\right\}<H\ \ \textit{for all}\ y\in\mathbb{R}^3. \tag{25}$$

Proof. In fact, according to $dimE^-<\infty$, by the equivalence of the norms in a finite-dimensional space, there exists $C_\xi>0$, which does not depend on y such that

$$C_\xi(\max\left\{\left|\xi_1(y)\right|,\ldots,\left|\xi_n(y)\right|\right\})^2\leq\|\sum_{i=1}^n\xi_i(y)\varphi_i(x)\|_{V_\infty}^2.$$

Hence,

$$\begin{aligned}\|w_0\|_{V_\infty}^2\geq\|w_0^-(\cdot-y)\|_{V_\infty}^2&=\|\sum_{i=1}^n\xi_i(y)\varphi_i(x)\|_{V_\infty}^2\\&\geq C_\xi(\max\left\{\left|\xi_1(y)\right|,\ldots,\left|\xi_n(y)\right|\right\})^2.\end{aligned}\tag{26}$$

This proves Claim 9 by choosing $H=\frac{\|w_0\|_{V_\infty}^2}{\sqrt{D}}>0$. ■ Then, substituting (24) in (25), we obtain $\|w_0^-(\cdot-y)\|^2<\varepsilon nH$ for $|y|\geq\hat{M}$. Since the norms $\|\cdot\|_{V_\infty}$ and $\|\cdot\|$ are equivalent in E, it follows that $\|w_0^-(\cdot-y)\|V_\infty\to0$ as $|y|\to\infty$. Thus,

$$\int_{\mathbb{R}^3}\left(w_0^-\right)^2(x-y)\mathrm{d}x\leq\frac{1}{V_\infty}\|w_0^-(\cdot-y)\|_{V_\infty}^2\to0\quad as\ \ |y|\to\infty, \tag{27}$$

concluding the proof of Claim 8. ■

By hypothesis (f_2), $|f(\cdot)|/|\cdot|$ is a bounded function in $\mathbb{R}\backslash\{0\}$ with $|f(\cdot)|/|\cdot|\leq m$. For (22), since w_0 is radial and continuous, the function $f(w_0(\cdot))/w_0(\cdot)$ assumes its maximum at $x_0\in\mathbb{R}^3$.

$$\begin{aligned}&\int_{\mathbb{R}^3}f\left(w_0(z)\right)w_0(z)\mathrm{d}z-m\int_{\mathbb{R}^3}w_0^2(z)\mathrm{d}z\\&\quad=\int_{\mathbb{R}^3}\left(\frac{f\left(w_0(z)\right)}{w_0(z)}-m\right)w_0^2(z)\mathrm{d}z\\&\quad\leq\left(\frac{f\left(w_0\left(x_0\right)\right)}{w_0\left(x_0\right)}-m\right)\|w_0\|_{L^2(\mathbb{R}^3)}^2\\&\quad\leq\frac{1}{2}\left(\frac{f\left(m-w_0\left(x_0\right)\right)}{w_0\left(x_0\right)}\right)\|w_0\|_{L^2(\mathbb{R}^3)}^2<-\zeta,\end{aligned}$$

where $\zeta = \frac{1}{2}\left(m - \frac{f(w_0(x_0))}{w_0(x_0)}\right) \|w_0\|^2_{L^2(\mathbb{R}^3)} > 0$, that is for (22), here exists $\zeta > 0$ such

$$\int_{\mathbb{R}^3} f\left(w_0(z)\right) w_0(z)\mathrm{d}z - m\int_{\mathbb{R}^3} w_0^2(z)\mathrm{d}z < -\zeta. \tag{28}$$

Substituting (24) and (27) in (20), we obtain

$$J_V(z) \leq \frac{1}{2}\|z\|^2\left\{\lambda^2\left[-\zeta + o_{|y|}(1)\right] + o_R(1)\right\} \tag{29}$$

for $|y|$ and R sufficiently large. To summarize the proof of this lemma, we can notice that we then have

$$\begin{aligned} J_V(z) &= \frac{1}{2}\|z\|^2\left(\lambda^2\left\|w_0^+(\cdot - y)\right\|^2 - \left\|v^-\right\|^2\right) - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_z z^2\mathrm{d}x - \int_{\mathbb{R}^3} F(z)\mathrm{d}x \\ &\leq \frac{1}{2}\|z\|^2\left(\lambda^2\left\|w_0^+(\cdot - y)\right\|^2\mathrm{d}x - \left\|v^-\right\|^2\right), \end{aligned} \tag{30}$$

here F is a nonnegative function by hypothesis (f_3). According to $\|\lambda w_0^+(\cdot - y)^2 + v^-\|^2 = 1$, it follows that $\|\lambda w_0^+(\cdot - y)\|^2 + \|v^-\|^2 = 1$. Moreover $z = \|z\|(w_0^+(\cdot - y) + v^-)$, the equation (30) can be write

$$\begin{aligned} J_V(z) &\leq \frac{1}{2}\|z\|^2\left(\lambda^2\left\|w_0^+(\cdot - y)\right\|^2 - \left\|v^-\right\|^2\right) \\ &= \frac{1}{2}\|z\|^2\left(2\lambda^2\left\|w_0^+(\cdot - y)\right\|^2 - 1\right). \end{aligned}$$

Thanks to the translation invariance of the norms and the equivalence of norm, it follows that $\|\cdot\|_{V_\infty}$, there exists $C_V > 0$, which does not depend on y, such that

$$2\|w_0^+(\cdot - y)\|^2 \leq C_V\|w_0\|_{V_\infty}.$$

For such values of λ, the lemma is proved

$$\lambda^2 < \frac{1}{C_V\|w_0\|_{V_\infty}} \leq \frac{1}{2\|w_0^+(\cdot - y)\|^2},$$

we have $J_V(w) < 0$. On the other hand, if $\lambda^2 \geq \frac{1}{(C_V\|w_0\|_{V_\infty})} := p_0 > 0$, it follows that $-\zeta + o_{|y|}(1) < -\frac{\zeta}{2}$ for $y \in \mathbb{R}^3$ with $|y|$ sufficiently large.
Hence, inequality (29) becomes

$$J_V(z) \leq \frac{1}{2}\|z\|^2\left[-\lambda^2\frac{\zeta}{2} + o_R(1)\right].$$

By $-\lambda^2 \leq -p_0$, choosing R sufficiently large such that $-p_0\frac{\zeta}{2} + o_R(1) \leq 0$, we obtain

$$J_V(z) \leq \frac{1}{2}\|z\|^2\left[-\lambda^2\frac{\zeta}{2} + o_R(1)\right] \leq \frac{1}{2}\|z\|^2\left[-p_0\frac{\zeta}{2} + o_R(1)\right] \leq 0$$

with the values $\zeta^2 \geq p_0$. then the proof of the lemma is complete. ■

4 Boundedness of a cerami sequence

Before we state the next result, we note that, if $\{v_n\}$ is a bounded sequence in E, then v_n satisfies one of the following cases:

(i) Vanishing: for all $r > 0$, limsup $\sup_{n\to\infty}\int_{B(y,r)}|v_n|^2\mathrm{d}x = 0$.
(ii) Or nonvanishing: there exist $r, \eta > 0$ and a sequence $(y_n) \subset \mathbb{R}^3$ such that $\lim_{n\to\infty}\int_{B(y_n,r)}|v_n|^2\mathrm{d}x > \eta$.

Lemma 10 *Assume that $\{u_n\} \subset E$ such that*

$$J_V(u_n) \to c \text{ and } \|J_V'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0 \quad \text{as } n \to \infty. \tag{31}$$

Then the sequence $\{u_n\}$ has a subsequence which is bounded in $H^1(\mathbb{R}^3)$.

Proof. Assume contradiction by suppose $\|u_n\| \to \infty$. Consider $v_n = \frac{u_n}{\|u_n\|}$, we deduce that the sequence $\{v_n\}$ is bounded with $\|v_n\| = 1$. But in fact, both (i) and (ii) are going to turn out to be wrong. First, suppose the hypothesis (i) hold for the sequence $\{v_n\}$. Recalling that the sequence (u_n) is a Cerami sequence, we have $J'_V(u_n)u_n^+ \to 0$ and $J'_V(u_n)u_n^- \to 0$. Therefore,

$$\begin{aligned} o_n(1) &= J'_V(u_n)\frac{u_n^+}{\|u_n\|^2} = \frac{1}{\|u_n\|}J'_V(u_n)v_n^+ \\ &= \left\|v_n^+\right\|^2 - \frac{1}{4\|u_n\|^2}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_nu_n^+\mathrm{d}x - \int_{\mathbb{R}^3}\left(\frac{f(u_n)}{u_n}v_nv_n^+\right)\mathrm{d}x \end{aligned} \tag{32}$$

and

$$\begin{aligned} o_n(1) &= J'_V(u_n)\frac{u_n^-}{\|u_n\|^2} = \frac{1}{\|u_n\|}J'_V(u_n)v_n^- \\ &= \left\|v_n^-\right\|^2 - \frac{1}{4\|u_n\|^2}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_nu_n^-\mathrm{d}x - \int_{\mathbb{R}^3}\left(\frac{f(u_n)}{u_n}v_nv_n^-\right)\mathrm{d}x. \end{aligned} \tag{33}$$

Subtracting the equation (42) from (32),

$$\begin{aligned} o_n(1) &= \left\|v_n^+\right\|^2 + \left\|v_n^-\right\|^2 - \frac{1}{4\|u_n\|^2}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n(u_n^- - u_n^-)\mathrm{d}x - \int_{\mathbb{R}^3}\left(\frac{f(u_n)}{u_n}v_n\left(v_n^+ - v_n^-\right)\right)\mathrm{d}x \\ &= \|v_n\|^2 - o_n(1) - \int_{\mathbb{R}^3}\left(\frac{f(u_n)}{u_n}v_n\left(v_n^+ - v_n^-\right)\right)\mathrm{d}x \\ &= 1 - \int_{\mathbb{R}^3}\left(\frac{f(u_n)}{u_n}v_n\left(v_n^+ - v_n^-\right)\right)\mathrm{d}x. \end{aligned}$$

One gets,

$$\int_{\mathbb{R}^3}\left(\frac{f(u_n)}{u_n}v_n\left(v_n^+ - v_n^-\right)\right)\mathrm{d}x \to 1. \tag{34}$$

By equivalence of the norms, there exist constants $C_1, \nu_0 > 0$ such that

$$\|z\| \le C_1\|z\|_{H^1(\mathbb{R}^3)} \le \nu_0\|z\|, \quad \text{for all functions } z \in E. \tag{35}$$

Given $0 < \varepsilon < \frac{1}{2}\nu_0$, and by hypothesis (f_1), there exists $\delta > 0$ such that

$$\frac{|f(l)|}{|l|} \le \varepsilon \quad for \quad 0 \ne |l| < \delta.$$

For each $n \in \mathbb{N}$, consider the set $\tilde{\Omega}_n = \left\{x \in \mathbb{R}^3 : |u_n| < \delta\right\}$. By following (35) and Hölder's inequality, we obtain

$$\begin{aligned} &\textstyle\int_{\tilde{\Omega}_n}\left(\frac{f(u_n)}{u_n}v_n\left(v_n^+ - v_n^-\right)\right)\mathrm{d}x \le \varepsilon\int_{\tilde{\Omega}_n}|v_n|\left|v_n^+ - v_n^-\right|\mathrm{d}x \\ &\le \varepsilon\left(\|v_n\|_{L^2(\mathbb{R}^N)}\|v_n^+\|_{L^2(\mathbb{R}^N)} + \|v_n\|_{L^2(\mathbb{R}^N)}\|v_n^-\|_{L^2(\mathbb{R}^N)}\right) \\ &\le 2\varepsilon\|v_n\|_{L^2(\mathbb{R}^N)} \le \tfrac{2\varepsilon}{\nu_0}\|v_n\|^2 = \tfrac{2\varepsilon}{\nu_0} < 1. \end{aligned}$$

Recall (34), we deduce that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3\backslash\tilde{\Omega}_n}\left(\frac{f(u_n)}{u_n}v_n\left(v_n^+ - v_n^-\right)\right)\mathrm{d}x > 0. \tag{36}$$

By Hölder's inequality with exponent $\frac{p}{2} > 1$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3/\tilde{\Omega}_n}v_n^3\left(v_n^+ - v_n^-\right)\mathrm{d}x &\le \left(\int_{\mathbb{R}^3/\tilde{\Omega}_n}1^{\frac{p}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^3/\tilde{\Omega}_n}\left(v_n^3(v_n^+ - v_n^-)\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{2}{p}} \\ &\le C\mu\left(\mathbb{R}^3\backslash\tilde{\Omega}_n\right)^{\frac{p-2}{p}}\left(\int_{\mathbb{R}^3/\tilde{\Omega}_n}\left(v_n^2(v_n^+ + v_n^-)(v_n^+ - v_n^-)\right)^{\frac{p}{2}}\mathrm{d}x\right)^{\frac{2}{p}} \\ &\le C\mu\left(\mathbb{R}^3\backslash\tilde{\Omega}_n\right)^{\frac{p-2}{p}}\|v_n\|^4_{L^p(\mathbb{R}^3)}. \end{aligned} \tag{37}$$

Since the function $|f(\cdot)|/|\cdot|$ is bounded, by (37), we obtain

$$\int_{\mathbb{R}^3/\tilde{\Omega}_n}\left(\frac{f(u_n)}{u_n}v_n^3\left(v_n^+-v_n^-\right)\right)\mathrm{d}x\leq C\mu\left(\mathbb{R}^3\backslash\tilde{\Omega}_n\right)^{(p-2)/p}\|v_n\|^4_{L^p(\mathbb{R}^3)}.$$

Assumption (i) and Lions's lemma ensure that $\|v_n\|_{L^p(\mathbb{R}^3)}\to 0$. Therefore, up to a subsequence, from (36), we obtain

$$\mu(\mathbb{R}^3/\tilde{\Omega}_n)\to\infty. \tag{38}$$

Now, we consider two disjoint subsets of $\mathbb{R}^3/\tilde{\Omega}_n$. Hypothesis (f_3) implies there exists $R>0$ such that, if $|l|>R$,

$$\frac{1}{4}f(l)l-F(l)>1.$$

Without loss of generality, we assume $0<\delta<R$, for each $n\in N$, we let $P_n:=\left\{x\in\mathbb{R}^3:|u_n(x)|>R\right\}$, and thus

$$c+o_n(1)\geq\int_{P_n}\left(\frac{1}{4}f(u_n(x))u_n(x)-F(u_n(x))\right)\mathrm{d}x\geq\mu(P_n),$$

which implies that the sequence $\{\mu(P_n)\}$ is bounded. We consider then also $Q_n:=\left\{x\in\mathbb{R}^3:\delta\leq|u_n(x)|\leq R\right\}$. Since $Q_n=(\mathbb{R}^3\backslash\tilde{\Omega}_n)\backslash P_n$, we have

$$\mu(\mathbb{R}^3/\tilde{\Omega}_n)=\mu(P_n)+\mu(Q_n).$$

It follows from (38) and the boundedness of the sequence $\mu(P_n)$ that

$$\mu(Q_n)\to\infty. \tag{39}$$

Since the interval $[\delta,R]$ is compact and the functions f and F are continuous, we have by hypothesis (f_3) that $\tilde{\delta}:=inf_{l\in[\delta,R]}(\frac{1}{2}f(l)l-F(l))>0$. Thus, from (39),

$$\int_{\mathbb{R}^3}\left(\frac{1}{4}f(u_n(x))u_n(x)-F(u_n(x))\right)\mathrm{d}x\geq\int_{Q_n}\left(\frac{1}{4}f(u_n(x))u_n(x)-F(u_n(x))\right)\mathrm{d}x\geq\tilde{\delta}\mu(B_n)\to\infty.$$

Again, we have a contradiction with the fact that

$$\begin{aligned}\int_{\mathbb{R}^3}\left(\frac{1}{4}f(u_n(x))u_n(x)-F(u_n(x))\right)\mathrm{d}x&=J_V(u_n)-\frac{1}{4}J'_V(u_n)u_n-\frac{1}{4}\int_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2\mathrm{d}x\\&\leq c+o_n(1).\end{aligned}$$

Notice that hypothesis (i) is not true. On the other hand, we suppose that (ii) holds for the sequence $\{v_n\}$. Write $f(l)=ml+(f(l)-ml)=ml+f_\infty(l)$. Let $\{y_n\}\subset\mathbb{R}^3$ be the sequence given by hypothesis (ii). Considering $\varphi_n(x)=\varphi_n(x-y_n)$ for any $\varphi\in C_0^\infty(\mathbb{R}^3)$, by (35) we have

$$\begin{aligned}|J'_V(u_n)\varphi_n|&\leq\|J'_V(u_n)\|_{E^*}\|\varphi_n\|\leq C_1\|J'_V(u_n)\|_{E^*}\|\varphi_n\|_E\\&=C_1\|J'_V(u_n)\|_{E^*}\|\varphi\|_E\to 0.\end{aligned}$$

Since $\|u_n\|\to\infty$, the measure of the set $A_n:=\left\{x\in\mathbb{R}^3:|u_n(x)|\neq 0\right\}$ is positive. Therefore,

$$\begin{aligned}o_n(1)=\frac{1}{\|u_n\|}J'_V(u_n)\varphi_n&=\left\langle v_n^+-v_n^-,\varphi_n\right\rangle-\int_{\mathbb{R}^3}K(x)\phi_{u_n}v_n\varphi_n\mathrm{d}x-\int_{\mathbb{R}^3}\frac{f(u_n)}{\|u_n\|}\varphi_n\mathrm{d}x\\&=\left\langle v_n^+-v_n^-,\varphi_n\right\rangle-\int_{\mathbb{R}^3}K(x)\phi_{u_n}v_n\varphi_n\mathrm{d}x-\int_{\mathbb{R}^3}mv_n\varphi_n\mathrm{d}x-\int_{\mathbb{R}^3}\frac{f_\infty(u_n)}{\|u_n\|}\varphi_n\mathrm{d}x\\&=\left\langle v_n^+-v_n^-,\varphi_n\right\rangle-\int_{\mathbb{R}^3}K(x)\phi_{u_n}v_n\varphi_n\mathrm{d}x-\int_{\mathbb{R}^3}mv_n\varphi_n\mathrm{d}x-\int_{\Omega_n}\frac{f_\infty(u_n)}{u_n}v_n\varphi_n\mathrm{d}x.\end{aligned} \tag{40}$$

Setting $\widetilde{v}_n(x) := v_n(x+y_n)$ and $\widetilde{u}_n := (x)u_n(x+y_n)$. Because of (V_4), $K(x) \to 0$ and hypothesis $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_n} v_n \varphi_n \mathrm{d}x &= \int_{\mathbb{R}^3} K(x+y_n)\phi_{\widetilde{u}_n}\widetilde{v}_n\varphi(x)\mathrm{d}x \\ &= \int_{|x|\le R} K(x+y_n)\phi_{\widetilde{u}_n}\widetilde{v}_n\varphi(x)\mathrm{d}x + \int_{|x|>R} K(x+y_n)\phi_{\widetilde{u}_n}\widetilde{v}_n\varphi(x)\mathrm{d}x \\ &\le \max K(x+y_n)\int_{|x|\le R}\phi_{\widetilde{u}_n}\widetilde{v}_n\varphi(x)\mathrm{d}x + \int_{|x|>R} K(x+y_n)\phi_{\widetilde{u}_n}\widetilde{v}_n\varphi(x)\mathrm{d}x \to 0.\end{aligned} \tag{41}$$

Notice that $\{\widetilde{v}_n\}$ is bounded in E. In fact, from (35) it follows that

$$\|\widetilde{v}_n\| \le C_1 \|\widetilde{v}_n\|_{H^1(\mathbb{R}^3)} = C_1 \|v_n\|_{H^1(\mathbb{R}^3)} \le \nu_0 \|v_n\| =: \nu_0.$$

One obtains up to a subsequence,

$$\begin{cases} \widetilde{v}_n \rightharpoonup \widetilde{v} & \text{in } E, \\ \widetilde{v}_n \to \widetilde{v} & \text{in } L^2_{\text{loc}}\left(\mathbb{R}^3\right). \end{cases} \tag{42}$$

Let $K = supp(\varphi)$. From (42) and (f_2), there exists a function $h \in L^1(K)$ such that $|\widetilde{v}_n(x)| \le h(x)$ almost everywhere in K. Thus, recalling that $f_\infty(x) = f(s) - ms$, we obtain

$$\left|\frac{f_\infty(\widetilde{u}_n)}{\tilde{u}_n}\widetilde{v}_n\varphi\right| \le 2mh(x)\varphi \in L^1(K). \tag{43}$$

We note that $\widetilde{v} \ne 0$, since by (ii) and (42),

$$\int_{B(0,r)} |\widetilde{v}|^2 \mathrm{d}x = \limsup_{n\to\infty}\int_{B(0,r)} |\widetilde{v}_n|^2 \mathrm{d}x = \limsup_{n\to\infty}\int_{B(y_n,r)} |v_n|^2 \mathrm{d}x > \eta > 0.$$

By hypothesis (f_1), $f_\infty(l)/l \to 0$ if $|l| \to \infty$, and from (43) and the Lebesgue Dominated Convergence Theorem, it follows that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{f_\infty(u_n)}{u_n} v_n\varphi_n \mathrm{d}x &= \int_K \frac{f_\infty(\tilde{u}_n)}{\tilde{u}_n}\tilde{v}_n\varphi\mathrm{d}x \\ &= \int_K \frac{f_\infty(\tilde{v}_n\|\tilde{u}_n\|)}{\tilde{v}_n\|\tilde{u}_n\|}\tilde{v}_n\varphi\mathrm{d}x \to 0.\end{aligned} \tag{44}$$

Thus, from (40), (42), (44), and the Change of Variables Theorem, one has

$$\begin{aligned} o_n(1) =& \frac{1}{\|u_n\|} J_V'(u_n)\varphi_n \\ =& \left\langle v_n^+ - v_n^-, \varphi_n\right\rangle - \int_{\mathbb{R}^3} K(x)\phi_{u_n} v_n\varphi_n\mathrm{d}x - \int_{\mathbb{R}^3} m v_n\varphi_n\mathrm{d}x - \int_{\mathbb{R}^3}\frac{f_\infty(u_n)}{\|u_n\|}\varphi_n\mathrm{d}x \\ =& \int_{\mathbb{R}^3}\left(\nabla v_n^+\nabla\varphi(x-y_n) + V(x)v_n^+\varphi(x-y_n)\right)\mathrm{d}x \\ &+ \int_{\mathbb{R}^3}\nabla v_n^-\nabla\varphi(x-y_n)\mathrm{d}x + \int_{\mathbb{R}^3} V(x)v_n^-\varphi(x-y_n)\mathrm{d}x - \int_{\mathbb{R}^3} m v_n\varphi_n\mathrm{d}x - o_n(1) \\ =& \int_{\mathbb{R}^3}\left(\nabla\widetilde{v}_n^+\nabla\varphi + V(x+y_n)\widetilde{v}_n^+\varphi\right)\mathrm{d}x \\ &+ \int_{\mathbb{R}^3}\left(\nabla\widetilde{v}_n^-\nabla\varphi + V(x+y_n)\tilde{v}_n^-\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^3} m\widetilde{v}_n\varphi dx.\end{aligned} \tag{45}$$

Case 1. $|y_n| \to \infty$. In this case, hypothesis (V_2) ensures that $V(x+y_n)$ converges to V_∞ almost everywhere in $\mathbb{R}^3$ as $n \to \infty$. From (38),

$$\begin{aligned} o_n(1) =& \int_{\mathbb{R}^3}\left(\nabla\widetilde{v}_n^+\nabla\varphi + (V_\infty + o_n(1))\tilde{v}_n^+\varphi\right)\mathrm{d}x \\ &+ \int_{\mathbb{R}^3}\left(\nabla\widetilde{v}_n^-\nabla\varphi + (V_\infty + o_n(1))\tilde{v}_n^-\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^3} m\widetilde{v}_n\varphi dx.\end{aligned} \tag{46}$$

Letting $n \to \infty$ in (39) and remembering that (35) holds, then for any function $\varphi \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3} \left(\nabla(\widetilde{v}_n^+ + \widetilde{v}_n^-)\right) \nabla\varphi + V_\infty(\widetilde{v}_n^+ + \widetilde{v}_n^-)\varphi \mathrm{d}x - \int_{\mathbb{R}^3} m\widetilde{v}_n\varphi dx = 0,$$

that is, $\widetilde{v} \neq 0$ is a weak solution of the problem $-\triangle\widetilde{v} + V_\infty\widetilde{v} = m\widetilde{v}$ in $\mathbb{R}^3$. Since $\widetilde{v} \neq m$ and there is no eigenfunction of the Laplacian in $\mathbb{R}^3$, his is a contradiction. Therefore, (ii) is not true in this case.

$Case$ 2. For $\{y_n\}$ is a bounded sequence, it follows from (35) that,

$$\|\tilde{u}_n\| \geq \frac{C_1}{\nu_0} \|\tilde{u}_n\|_{H^1(\mathbb{R}^3)} = \frac{C_1}{\nu_0} \|u_n\|_{H^1(\mathbb{R}^3)} \geq \frac{1}{\nu_0} \|u_n\|,$$

which goes to infinity as $n \to \infty$, and

$$0 \neq |\widetilde{v}(x)| = \lim_{n\to\infty} |\widetilde{v}_n(x)| = \lim_{n\to\infty} \frac{|\widetilde{u}_n(x)|}{|\widetilde{u}_n|} \text{ almost everywhere in } \Omega$$

with $\mu(\Omega) > 0$ and $\Omega \subset B(0, r)$. By following $\|\widetilde{u}_n\| \to \infty$, we have $\widetilde{u}_n(x) \to \infty$ almost everywhere in Ω. Thus, Fatous Lemma and hypothesis (f_3) yield

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\mathbb{R}^3} \left[\frac{1}{2} f(u_n)u_n - F(u_n)\right] \mathrm{d}x \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^3} \left[\frac{1}{2} f(\widetilde{u}_k)\widetilde{u}_k - F(\widetilde{u}_k)\right] \mathrm{d}x \\
&\geq \liminf_{n\to\infty} \int_{\Omega} \left[\frac{1}{2} f(\widetilde{u}_k)\widetilde{u}_k - F(\widetilde{u}_k)\right] \mathrm{d}x \\
&\geq \int_{\Omega} \liminf_{n\to\infty} \left[\frac{1}{2} f(\widetilde{u}_k)\widetilde{u}_k - F(\widetilde{u}_k)\right] \mathrm{d}x \\
&= \infty.
\end{aligned}$$

However,

$$J_V(u_n) - \frac{1}{2} J_V'(u_n)u_n = \int_{\mathbb{R}^3} \left(\frac{1}{2} f(u_n)u_n - F(u_n)\right) \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 \mathrm{d}x = c + o_n(1). \tag{47}$$

Hence,

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} f(u_n)u_n - F(u_n)\right) \mathrm{d}x = J_V(u_n) - \frac{1}{2} J_V'(u_n)u_n - \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 \mathrm{d}x \leq c + o_n(1).$$

One can deduce a contradiction with $Case$ 2 for the sequence $\{y_n\}$. Therefore, (ii) does not hold either for the sequence $\{v_n\}$. We conclude that, up to a subsequence, $\{u_n\}$ is bounded. ■

Next we want to show the compactness of a Cerami sequence of the functional J_V, we use the Concentration-compactness Principle of Lions, which was proved in [19].

Lemma 11 *It holds that*

$$c < c_\infty := \inf \left\{ J_\infty(z) : z \in H^1\left(\mathbb{R}^N\right) \setminus \{0\}, J_\infty'(z) = 0 \right\}.$$

To prove this lemma, in what follows, we need some auxiliary lemmas. The first two may be refer to [1].

Lemma 12 *There exists $\nu \in (1, 2]$ with the following property: for any given $C_3 \geq 1$ there is a constant $C_4 \geq 0$ such that the inequalities*

$$|f(u + v) - f(u) - f(v)| \leq C_4|uv|^{\frac{\nu}{2}},$$

$$|F(u + v) - F(u) - F(v) - f(u)v - f(v)u| \leq C_4|uv|^\nu,$$

hold true for all $u, v \in \mathbb{R}$ with $|u|, |v| \leq C_3$.

Proof. Set $p := p_1$ and $\nu := \min\left\{\frac{1}{2}(p+1), 2\right\}$. Observe that (f_4) implies that $|f^k(u)| \le C|u|^{p-k}$ if $0 < |u| \le C_3$. The proof of the above inequalities for $f(u) = |u|^{p-1}u$ istedious but elementary. We reduce the general case to this special case as follows: Let $u, v > 0$. Then

$$\begin{aligned}|f(u+v) - f(u) - f(v)| &= \left|\int_0^u \int_r^{r+v} f''(s) ds\, dr\right| \\ &\le C \int_0^u \int_r^{r+v} s^{p-2} ds\, dr \\ &\le C\left((u+v)^p - u^p - v^p\right) \le C(uv)^{\frac{\nu}{2}},\end{aligned}$$

$$\begin{aligned}|F(u+v) - F(u) - F(v) - f(u)v - f(v)u| &= \left|\int_0^u \int_0^v \int_0^r \int_t^{s+l} f'''(w) dw\, dl\, dr\, ds\right| \\ &\le C \int_0^u \int_0^v \int_0^r \int_t^{s+l} w^{p-3} dw\, dl\, dr\, dt \\ &\le C\left|(u+v)^{p+1} - u^{p+1} - v^{p+1} - u^p v - v^p u\right| \\ &\le C(uv)^{\nu}.\end{aligned}$$

The other cases are similar. ■

Lemma 13 *If $\nu_2 > \nu_1 \ge 0$, there exists $C_V > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^3$,*

$$\int_{\mathbb{R}^3} e^{-\nu_1|x-x_1|} e^{-\nu_2|x-x_2|} \mathrm{d}x \le C_V e^{-\nu_1|x_1-x_2|}.$$

Proof. Since

$$\begin{aligned}&\nu_1|x_1 - x_2| + (\nu_2 - \nu_1)|x - x_2| \\ &\quad\le \nu_1\left(|x - x_1| + |x - x_2|\right) + (\nu_2 - \nu_1)|x - x_2| \\ &\quad= \nu_1|x - x_1| + \nu_2|x - x_2|.\end{aligned}$$

We have

$$\begin{aligned}\int_{\mathbb{R}^3} e^{-\nu_1|x-x_1|} e^{-\nu_2|x-x_2|} dx &\le \int_{\mathbb{R}^3} e^{-\nu_1|x_1-x_2|} e^{-(\nu_2-\nu_1)|x-x_2|} dx \\ &= C_V e^{-\nu_1|x_1-x_2|},\end{aligned}$$

and the lemma follows. ■

We note that the set M defined in the Theorem 3.1 is closed and bounded, and is contained in a finite-dimensional space, namely, in the space $\mathbb{R}w_0(\cdot - y) \oplus E^-$. Therefore, M is a compact set, which implies that for all $y \in \mathbb{R}^3$, there exists $z_y = v_y^- + s_y w_0^+(\cdot - y) \in M$ satisfying

$$\max_{z \in M} J_V(z) = J_V(v_y^- + s_y w_0^+(\cdot - y)),$$

since J_V is a continuous functional. The following result shows that the values s_y are uniformly bounded on y by positive constants if $|y|$ is sufficiently large.

Lemma 14 *There are $A, B \in R$, not depending on y, so that $0 < A \le s_y \le B$ for $|y|$ is sufficiently large.*

Proof.

Since $z_y = v_y^- + s_y w_0^+(\cdot - y) \in M$, here the number $R > 0$ is given by Lemma 7 and does not depend on y, one has

$$R^2 \ge \|z_y\|^2 = \left\|v_y^-\right\|^2 + s_y^2\left\|w_0^+(\cdot - y)\right\|^2 \ge s_y^2(\|w_0(\cdot - y)\|^2 - \left\|w_0^-(\cdot - y)\right\|^2).$$

As proven in (27), we can take $|y|$ large enough to ensure that

$$\left\|w_0^-(\cdot - y)\right\|^2 \le \frac{C_V}{2}\left\|w_0\right\|_{V_\infty}^2,$$

where $C_V > 0$ does not depend on y and satisfies $\|w_0(\cdot - y)\|^2 \geq C_V \|w_0\|^2_{V_\infty}$. Thus,

$$\begin{aligned} R^2 \geq \|z_y\|^2 &= \left\|v_y^-\right\|^2 + t_y^2 \left\|w_0^+(\cdot - y)\right\|^2 \\ &\geq s_y^2 \left\|w_0^+(\cdot - y)\right\|^2 \\ &\geq s_y^2 \left(\|w_0(\cdot - y)\|^2 - \left\|w_0^-(\cdot - y)\right\|^2\right) \\ &\geq s_y^2 \left(C_V \|w_0\|^2_{V_\infty} - \frac{C_V}{2} \|w_0\|^2_{V_\infty}\right) \\ &= \frac{s_y^2 C_V}{2} \|w_0\|^2_{V_\infty}. \end{aligned}$$

That is, we have $s_y^2 \leq 2R^2 / \left(C \|w_0\|^2_{V_\infty}\right) := B^2$. Thus, we take $t_0 > 0$, which does not depend on y, sufficiently small so that $\|t_0 w_0^+(\cdot - y)\| \leq \rho < R$. By following the Remark 5, we conclude that $I(t_0 w_0^+(\cdot - y)) \geq \alpha > 0$. Consequently,

$$J_V(v_y^- + s_y w_0^+(\cdot - y)) = \max_{z \in M} J_V(z) \geq J_V(s_0 w_0^+(\cdot - y)) \geq \alpha,$$

that is to say,

$$\begin{aligned} &\tfrac{s_y^2}{2} \left\|w_0^+(\cdot - y)\right\|^2 - \tfrac{1}{2} \|v_y\|^2 - \textstyle\int_{\mathbb{R}^3} F\left(v_y^- + s_y w_0^+(x - y)\right) \mathrm{d}x \\ &= J_V\left(v_y^- + s_y w_0^+(\cdot - y)\right) \geq \alpha. \end{aligned}$$

Therefore, since F is nonnegative,

$$\frac{s_y^2}{2} \left\|w_0^+(\cdot - y)\right\|^2 \geq \alpha.$$

This shows that

$$s_y^2 \geq \frac{2\alpha}{C\|w_0\|^2_{V_\infty}} := A^2,$$

where $C > 0$ does not depend on $\|y\|$ and satisfies$\|w_0^+(\cdot - y)\|^2 \leq C\|w_0\|^2_{V_\infty}$. The lemma is thus proved. ■

In the following, we present the proof of Lemma 11.

Proof of Lemma 11. We denote $w_{0,y}(x) := w_0(x - y)$. Since F is nonnegative and $C > 0$ is a constant, recalling the definitions of the functionals J_V and J_∞, we have

$$\begin{aligned} &J_V(v_y^- + s_y w_{0,y}^+) \\ &= \frac{s_y^2}{2}\|w_{0,y}^+\|^2 - \frac{1}{2}\|v_y^-\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{v_y^- + s_y w_{0,y}^+}(v_y^- + s_y w_{0,y}^+)^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v_y^- + s_y w_{0,y}^+(x))\mathrm{d}x \\ &\leq \frac{s_y^2}{2}\|w_{0,y}\|^2 - \int_{\mathbb{R}^3} F(s_y w_{0,y})\mathrm{d}x - \frac{s_y^2}{2}\left\|w_{0,y}^-\right\|^2 + \int_{\mathbb{R}^3} (F(s_y w_{0,y}) - F(v_y^- + t_y u_{0,y}^+))\mathrm{d}x \\ &\leq J_\infty(s_y w_{0,y}) + \frac{t_y^2}{2}\int_{\mathbb{R}^3} (V(x) - V_\infty) w_{0,y}^2(x)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} (F(v_y^- - s_y w_{0,y}^-) + F(s_y w_{0,y}) - F(v_y^- + s_y w_{0,y}^+))\mathrm{d}x. \end{aligned} \tag{48}$$

Let us estimate the last integral in the above inequality. Taking $z_y^- = v_y^- - s_y w_{0,y}$, we want to estimate

$$\mathcal{I}_y := \left|\int_{\mathbb{R}^3} F(z_y^-) + F(s_y w_{0,y}(x)) - F(z_y^- + s_y w_{0,y}(x))\mathrm{d}x\right|.$$

It holds that

$$\begin{aligned} \mathcal{I}_y \leq &\int_{\mathbb{R}^3} |F\left(z_y^-\right) + F\left(s_y w_{0,y}\right) - F\left(z_y^- + s_y w_{0,y}\right) \\ &- f\left(z_y^-\right) s_y w_{0,y} - f\left(s_y w_{0,y}\right) z_y^-|\mathrm{d}x \\ &+ s_y \int_{\mathbb{R}^3} |f\left(z_y^-\right)| \, |w_{0,y}| \,\mathrm{d}x + \int_{\mathbb{R}^3} |f\left(s_y w_{0,y}\right)| \, |z_y^-|\mathrm{d}x. \end{aligned}$$

Taking $w_y^- \in M$ and $\|w_y^-\|^2 \leq R^2$, we may repeat the estimates in (26) with w_y^- in the place of $w_{0,y}^-$. By using Lemma 14, there is a constant $C > 0$ which does not depend on y such that

$$\begin{aligned} |w_y^-(x)| &= |\sum_{i=1}^{k} \xi_i(y)\varphi_i(x)| \\ &\leq \sum_{i=1}^{k} |\varphi_i(x)| \max\{|\xi_1(y)|, \ldots, |\xi_k(y)|\} \\ &\leq C\sum_{i=1}^{k} |\varphi_i(x)| \leq C\sum_{i=1}^{k} \sup|\varphi_i(x)| := D \end{aligned} \tag{49}$$

for all $x \in \mathbb{R}^3$. Without loss of generality, we may suppose that $D > 1$ also satisfies $|w_{0,y}(x)| \leq D$ for all x $\in \mathbb{R}^3$ since $w_0, y \in L^\infty(\mathbb{R}^3)$. The Lemma 14 and the hypothesis (f_2) are devoted to obtain a constant $C > 0$ such that

$$\mathcal{I}_\mathcal{Y} \leq C_4 t_\mathcal{Y}^\nu \int_{\mathbb{R}^3} |w_y^-|^\mu \cdot |w_{0,y}|^\nu \mathrm{d}x + 2mt_y \int_{\mathbb{R}^3} \left|w_y^-\right| |w_{0,y}| \,\mathrm{d}x, \tag{50}$$

where $\nu > 1$ is given by Lemma 12. Now, taking $\xi = \lambda_i < 0 < V^\infty$ in Theorem 3.19 from [28], it holds that any eigenfunction $\varphi_i,\ i = 1, ..., k$, satisfies

$$|\varphi_i(x)| \leq Ce^{-\delta|x|} \ \ for\ all\ x \in\ \mathbb{R}^3$$

for some $\sqrt{V_\infty} < \delta < \sqrt{V_\infty - \lambda_k}$. Thus, from the first inequality in (50), one has for $|y|$ sufficiently large that

$$|z_y^-(x)| \leq Ce^{-\delta|x|} \ \ for\ all\ x \in\ \mathbb{R}^3.$$

Since w_0 is a positive radial solution of problem (P_∞), given by Berestycki and Lions in [4], we have $|w_0(x)| \leq Ce^{-\delta|x|}$ for all $x \in \mathbb{R}^3$. It follows from Lemma 13 that

$$\int_{\mathbb{R}^3} \left|z_y^-\right| |w_{0,y}| \,\mathrm{d}x \leq C\int_{\mathbb{R}^3} e^{-\delta|x|} e^{-\sqrt{V_\infty}|x-y|}\mathrm{d}x \leq Ce^{-\sqrt{V_\infty}|y|}. \tag{51}$$

Analogously, by Lemma 13,

$$\begin{aligned} \int_{\mathbb{R}^3} \left|z_y^-\right|^\nu |u_{0,y}|^\nu \,\mathrm{d}x &\leq C\int_{\mathbb{R}^N} e^{-\delta\nu|x|} e^{-\nu\sqrt{V_\infty}|x-y|}\mathrm{d}x \\ &\leq Ce^{-\nu\sqrt{V_\infty}|y|} \leq Ce^{-\sqrt{V_\infty}|y|}, \end{aligned} \tag{52}$$

where $\nu > 1$. Since s_y is uniformly bounded by Lemma 14, by estimates (51) and (52) applied in (50), yield

$$\mathcal{I}_y \leq Ce^{-\sqrt{V_\infty}|y|}, \tag{53}$$

where the constant $C > 0$ does not depend on y. Now, let us estimate the integral

$$\frac{s_y^2}{2}\int_{\mathbb{R}^3} \left(V(x) - V_\infty\right) w_{0,y}^2(x)\mathrm{d}x. \tag{54}$$

By hypothesis (V_5), it follows that

$$\begin{aligned} \frac{s_y^2}{2}\int_{\mathbb{R}^3} \left(V(x) - V_\infty\right) w_{0,y}^2(x)\mathrm{d}x &= \frac{s_y^2}{2}\int_{\mathbb{R}^3} \left(V(x+y) - V_\infty\right) w_0^2(x)\mathrm{d}x \\ &\leq -Ce^{-\zeta|y|}, \end{aligned} \tag{55}$$

for $|y|$ sufficiently large, where $C > 0$ does not depend on y. Thus, follows from (53) that (48), it may be written as

$$J_V\left(v_y^- + s_y w_{0,y}^+\right) \leq J_\infty\left(s_y w_{0,y}\right) - Ce^{-\zeta|y|} + Ce^{-\sqrt{V_\infty}|y|}.$$

Therefore, by hypothesis (V_5), $0 < \zeta < \sqrt{V_\infty}$, we obtain

$$J_V\left(v_y^- + s_y w_{0,y}^+\right) < \max_{s\geq 0} J_\infty(sw_0), \tag{56}$$

where $|y|$ sufficiently large. The assumption (f_5) then ensures that the maximum $\max_{s\geq 0} J_V(sw_0)$ is attained exactly at $t=1$, because w_0 is a nontrivial critical point of problem (P_∞). This is the only moment in this paper where hypothesis (f_5) is needed. Since w_0 is a ground state solution for (P_∞), it follows from the definition of the value $c > 0$ that

$$c \leq \max_{z\in M} J_V(z) = J_V\left(v_y^- + s_y w_{0,y}^+\right) < c_\infty, \tag{57}$$

and the lemma is proved. ■

Proof of Theorem 1. By Theorem 3, functional J_V has a Cerami sequence $\{u_n\}$ at level $c > 0$. Up to a subsequence, $\{u_n\}$ is bounded by Lemma 10. Therefore, $u_n \rightharpoonup u$ for some $u \in H^1(\mathbb{R}^3)$. In the process of proving the convergence of a bounded Cerami sequence, we use the Concentration-compactness Principle of Lions to get the compactness. In fact, up to a subsequence, $u_n \to u$ strongly in $H^1(\mathbb{R}^3)$. Since J_V is a C^1functional and $J_V(u) = c > 0$, we have $u \in H^1(\mathbb{R}^3)$ is a nontrivial weak solution of problem (1). The proof of the theorem is thus completed. ■

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