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The Existence of Solutions for Nonlocal Elliptic Equations on Bounded Domain

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Abstract: This paper discusses the existence of solutions for the elliptic equations with nonlocal terms on bounded domain Ω. Under the condition of one mass constraint, different types of solutions about equations are found in subcritical, critical and supercritical cases. Due to the energy functional is coercive in subcritical or critical cases, the minimum principle can be used to find global solution. However, because the lack of coercive about functional in the supercritical case, this method is no longer applicable, so considering use the Mountain Pass Lemma to search for local solutions.

Keywords: Gagliardo-Nirenberg inequality; Hardy-Littlewood-Sobolev inequality; Minimum Principle; Mountain Pass Lemma

1 Introduction

People have great interest in nonlinear Schrödinger equations in all days. The discovery of each equation will lead to the fluctuation of mathematics. In 1928, D. R. Hartree put forward the Hartree equation, which regards each electron as moving in the average potential field provided by the rest electrons, and gives the equation of motion of each electron by iterative method. In 1930, B. A. Fock and J. C. Slater, the students of D. R. Hartree, respectively, put forward the self consistent field iterative equation considering Pauli principle, which is called Hartree Fock equation, further improving the Hartree equation developed by D. R. Hartree. In order to solve the Hartree Fock equation, in 1951, C. C. J. Roothaan further proposed that the molecular orbitals in the equation be expanded linearly by the atomic orbitals of the constituent molecules, and obtained the famous RHF equation. This equation and the method further developed on the basis of this equation are the fundamental methods of modern quantum chemistry.

In 1954, in one of Pekar's works, he described the quantum mechanics of the static polaron and proposed the classical stable Choquard-Pekar equation

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy \right) u, \ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

In 1976, Choquard used the classical stable Choquard-Pekar equation to simulate the trapped electrons in his own hole, which is to some extent similar to the Hartree Fock theory of one component plasma [1]. In 1977, Lieb continued to study the classical stable Choquard-Pekar equation. By using the symmetric rearrangement inequality and the maximum theory, the existence of the minimum value of the equation is proved. In addition, he uses comparison and scaling methods to prove the uniqueness of the minimum value [2]. In 1980, Lions et al. proved the existence of a series of radial symmetric solutions of above the classical stable Choquard-Pekar equation by using dual variational method [3].

In 2008, P.Choquard et al. continued to study the Choquard equation

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^{N-2}} dy \right) u, \ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{1}$$

When $N \geq 1$, they first simplify (1) to the corresponding second order differential equations, and then use the shooting method to prove the existence of the radial ground state solution. In addition, they proved the uniqueness of solution by

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studying decay characteristics [4]. In 2010, Ma and Zhao studied the more general Choquard equation

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|^\alpha} dy\right) |u|^{p-2}u,\ u \in H^1(\mathbb{R}^N),$$

where $\alpha \in (0, N)$, $2 \le \frac{1}{p} < \frac{2N-\alpha}{N-2}$, and making appropriate assumptions about p, α, n. They found that the positive solution of this equation is radially symmetric [5]. It is also proved that the positive solution is monotonically decreasing by the moving plane method studied by Chen et al [6]. In 2013, V. Moroz and J. Van Schaftingen studied the general Hartree type equation

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\right) |u|^{p-2}u,\ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $u > 0$, $u \in H^1(\mathbb{R}^N)$ and $N \ge 3$. In the best parameter range, they obtain the results of regularity, positive, radial symmetry and asymptotic of the ground state solution of the equation [7]. From the above equations, we can see that the biggest difference between them is the index. In terms of the existence and properties of solutions, the properties of solutions of these equations are relatively stable. With the development of the research, we are more interested in studying the properties of the solutions of the more general exponential equations.

In this paper, we mainly explore the existence of solutions of the following more general equation with nonlocal terms

$$-\Delta u(x) + \lambda u(x) = \mu \left(\int_{B_1} \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\right) |u(x)|^{p-2}u(x) \tag{2}$$

on bounded domain. We study the existence of solutions for equation (2) with one mass constraint

$$\begin{cases} -\Delta u(x) + \lambda u(x) = \mu \left(\int_\Omega \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\right) |u(x)|^{p-2}u(x) \\ u \in H_0^1(\Omega),\ u > 0,\ \int_\Omega u^2 dx = 1 \end{cases}. \tag{3}$$

In the case of subcritical, critical and supercritical, considering the problem of mandatory, we use two different methods to find solutions. Minimum Principle (subcritical and critical cases) and Mountain Pass Lemma [8] (supercritical case) be used in the proof. Besides, we construct the concrete form of the “best constant” ([9],[10]) of an interpolation estimate in critical case.

The structure of the paper is organized as follows. We begin in Section 2 with notations and some inequalities used throughout this paper. Section 3 is the main results of this article. Section 4 gives the specific expression form of the best constant in the critical situation. At last, Section 5 resolve the existence of solutions of problem (3) in the cases of subcritical, critical and supercritical.

2 Some notations and inequalities

Throughout the paper, we use the following notations:

- $\Omega \subset \mathbb{R}^N$ be a bounded domain;
- $||\cdot||_{H_0^1(\Omega)}$ is the norm of $H_0^1(\Omega)$ defined by $||u||^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2$;
- $||\cdot||_p$ is the norm of $L^p(\Omega)$ defined by $||u||_p = (\int_\Omega |u|^p)^{\frac{1}{p}}$ for $0 < p \le \infty$.

The following lemmas are the classical Hardy-Littlewood-Sobolev inequality [12] and Gagliardo-Nirenberg inequality.

Lemma 1 *(Hardy-Littlewood-Sobolev inequality). Let* $0 < t < N$, *and* $s, r > 1$ *be constants such that*

$$\frac{1}{r} + \frac{1}{s} + \frac{t}{N} = 2.$$

Assume that $f \in L^r(\mathbb{R}^N)$ *and* $g \in L^s(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)|x-y|^{-t} g(y) dx dy \le C(N, s, t)||f||_r ||g||_s. \tag{4}$$

From this classical Hardy-Littlewood-Sobolev inequality, we introduce the following inequality to the problem with nonlocal term.

Lemma 2 *(Gagliardo-Nirenberg inequality). Let $u \in L^p(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \le S\left(\int_{\mathbb{R}^N}|u|^2\right)^{\frac{(N+\alpha)-p(N-2)}{2}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{pN-(N+\alpha)}{2}}, \tag{5}$$

where $\alpha \in (0,N)$, $\frac{N-2}{N+\alpha} \le \frac{1}{p} \le \frac{N}{N+\alpha}$ and $S > 0$ is a constant.

Proof. Here, we give a briefly proof. For one thing, by the classical Hardy-Littlewood-Sobolev inequality, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \le C_1\left(\int_{\mathbb{R}^N}|u|^{\frac{2pN}{N+\alpha}}\right)^{\frac{N+\alpha}{N}}, \tag{6}$$

where $C_1 > 0$ is a constant. For another, by interpolation inequality, there exists a constant $C_2 > 0$ such that

$$\left(\int_{\mathbb{R}^N}|u|^{\frac{4N}{N+2}}\right)^{\frac{N+2}{4N}} \le \left(\int_{\mathbb{R}^N}|u|^2\right)^{\frac{\theta}{2}}\left(\int_{\mathbb{R}^N}|\nabla u|^{2^*}\right)^{\frac{1-\theta}{2}} \le C_2\left(\int_{\mathbb{R}^N}|u|^2\right)^{\frac{\theta}{2}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{1-\theta}{2}}, \tag{7}$$

where $2^* = \frac{2N}{N-2}(N \ge 3)$, $\theta = \frac{(N+\alpha)-p(N-2)}{2p}$. Combining (6) with (7), we get that (5) holds. ■

3 Main results

Inspired by [10], we generalize the main results to equations (3). It also can be rewritten as the following form:

$$\begin{cases}-\Delta u(x) + \lambda u(x) = \mu\phi_u(x)|u(x)|^{p-2}u(x)\\ u \in H_0^1(\Omega),\ u > 0,\ \int_\Omega u^2dx = 1\end{cases}. \tag{8}$$

In section 4, we give the concrete expression of the "best constant" in the case of $p = \frac{N+\alpha+2}{N}$ (see Remark 5). In section 5, in subcritical or critical cases, owing to the coercivity of the energy functional, the global solutions of problem (3) are found by minimization techniques. Nevertheless, the coercivity of the energy functional is vanished in supercritical case. Therefore, the previous method does not work. We consider to solve the supercritical case by using the Mountain Pass Lemma.

Theorem 3 *(subcritical, critical and supercritical cases: existence) Let $\lambda > 0$,*
$(H1)$ subcritical case: $\frac{N+\alpha}{N} \le p < \frac{N+\alpha+2}{N}$;
$(H2)$ critical case: $p = \frac{N+\alpha+2}{N}$, there are μ^* such that $0 < \mu < \mu^*$;
$(H3)$ supercritical case: $\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$.
Then the problem (3) admits at least one nontrivial solution. Moreover, the finding solutions are global in the condition of $(H1)$ and $(H2)$, which are local in $(H3)$.

4 The "best constant" as $p = \frac{N+\alpha+2}{N}$

First we construct the concrete form of the "best constant" about the interpolation inequality

$$\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \le C_{N,p}\left(\int_\Omega|\nabla u|^2\right)^{\frac{pN-(N+\alpha)}{2}}\left(\int_\Omega|u|^2\right)^{\frac{(N+\alpha)-p(N-2)}{2}} \tag{9}$$

in the critical case $p = \frac{N+\alpha+2}{N}$ for $0 < \alpha \le \frac{4}{N-2}$ and $N \ge 3$. The main purpose of this section is to propose the relationship between the best constant and the interpolation estimate. In order to calculate the best constant $C_{N,p}$, this is enough to minimize the functional

$$J_{\alpha,p}(u) = \frac{\left(\int_\Omega|\nabla u|^2\right)^{\frac{pN-(N+\alpha)}{2}}\left(\int_\Omega|u|^2\right)^{\frac{(N+\alpha)-p(N-2)}{2}}}{\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy} = \frac{||\nabla u||_2^{pN-(N+\alpha)}||u||_2^{(N+\alpha)-p(N-2)}}{\int_\Omega\phi_u\cdot|u|^pdx}. \tag{10}$$

First we give the following compactness lemma as a foreshadowing.
Compactness Lemma. $H^1_{rad}(\Omega) \hookrightarrow L^{2p}(\Omega)$ is a compact embedding for $1 < p < \frac{2^*}{2} = \frac{N}{N-2}$.
Proof. For $\forall u \in H^1(\Omega)$, based on the interpolation inequality

$$\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \le C||u||_{H^1}^{pN-(N+\alpha)}||u||_2^{(N+\alpha)-p(N-2)}.$$

If there is a bounded sequence $u_n \in H^1_{rad}(\Omega)$, such that $u_n \to 0$, as $n \to +\infty$, and then the compactness lemma will be proved. In fact, we can easily get this result from an estimate by Strauss's Radial Lemma [11]: If $u \in H^1_{rad}(\Omega)$, then $|u(x)| \le C|x|^{\frac{1-N}{2}}||u||_{H^1}$ for $|x| \ge 1$, where C depends only on N and $N \ge 3$. Compactness conclusion is obvious, we will not elaborate on the specific proof process here. ■

Lemma 4 *For* $1 < p < \frac{2^*}{2} = \frac{N}{N-2}$, $\gamma := \inf_{u \in H^1(\Omega)} J_{\alpha,p}(u)$ *is achieved by a function* φ *which possess the following properties:*

(i) $\varphi \in H^1(\Omega) \cap C^\infty(\Omega)$;

(ii) $\varphi > 0$ *and only depends on* $|x|$;

(iii) φ *is a ground state solution of the equation*

$$\frac{pN-(\alpha+N)}{2}\Delta\varphi - \frac{(N+\alpha)-p(N-2)}{2}\varphi + \int_\Omega \frac{|\varphi(y)|^p}{|x-y|^{N-\alpha}}dy|\varphi(x)|^{p-2}\varphi(x) = 0, \tag{11}$$

with minimal L^2 *norm. Moreover, we have* $\gamma = \frac{||\varphi||_2^{2(p-1)}}{p}$.

Proof. Let $u^{\eta,\kappa}(x) \equiv \kappa u(\eta x)$, by directly calculation, it is easy to get that
(a) $||u^{\eta,\kappa}||_2^2 = \eta^{-N}\kappa^2||u||_2^2$;
(b) $||\nabla_x u^{\eta,\kappa}||_2^2 = \eta^{2-N}\kappa^2||\nabla u||_2^2$;
(c) $J_{\alpha,p}(u^{\eta,\kappa}(x)) = J_{\alpha,p}(u)$.

Owing to $J_{\alpha,p}(u) \ge 0$, for $1 < p < \frac{N}{N-2}$, there are a minimizing sequence $u_n \in H^1(\Omega) \cap L^{2p}(\Omega)$ such that $\gamma = inf J_{\alpha,p}(u) = \lim_{n\to\infty} J_{\alpha,p}(u_n) < \infty$. Let us suppose u_n are positive and take $u_n = u_n(|x|)$ according to symmetrization property.

Taking $\eta_n = ||u_n||_2/||\nabla u_n||_2$ and $\kappa_n = ||u_n||_2^{\frac{N}{2}-1}/||\nabla u_n||_2^{\frac{N}{2}}$, we can find a sequence $\varphi_n(x) = u^{\eta_n,\kappa_n}(x)$ satisfy the following properties:
(a) $\varphi_n \in H^1(\Omega)$;
(b) $\varphi_n \ge 0$, $\varphi_n = \varphi_n(|x|)$;
(c) $||\varphi_n||_2 = ||\nabla\varphi_n||_2 = 1$;
(d) $J_{\alpha,p}(\varphi_n) \to \gamma$ as $n \to \infty$.

From $||\nabla\varphi_n||_2 = 1$, we know that φ_n is a bounded sequence in $H^1(\Omega)$, there are some weakly convergence subsequence in $H^1(\Omega)$ and record the limit as φ^*. From $\varphi_n = \varphi_n(|x|)$, it is easy to see that the sequence φ_n is radial in $H^1(\Omega)$. Due to the Compactness Lemma, it will be $\varphi_n \to \varphi^*$ are strongly in $L^{2p}(\Omega)$ for $1 < p < \frac{N}{N-2}$. From the weakly convergence, we have

$$||\varphi^*||_2 \le 1 \text{ and } ||\nabla\varphi^*||_2 \le 1.$$

Thus, we can get

$$\gamma \le J_{\alpha,p}(\varphi^*) = \frac{||\nabla\varphi^*||_2^{pN-(N+\alpha)}||\varphi^*||_2^{(N+\alpha)-p(N-2)}}{\int_\Omega\int_\Omega \frac{|\varphi^*(x)|^p|\varphi^*(y)|^p}{|x-y|^{N-\alpha}}dxdy} \le \frac{1}{\int_\Omega\int_\Omega \frac{|\varphi^*(x)|^p|\varphi^*(y)|^p}{|x-y|^{N-\alpha}}dxdy} = \lim_{n\to\infty} J_{\alpha,p}(\varphi_n) = \gamma.$$

That implies $||\nabla\varphi^*||_2^{pN-(N+\alpha)}||\varphi^*||_2^{(N+\alpha)-p(N-2)} = 1$ and $||\varphi^*||_2 = ||\nabla\varphi^*||_2 = 1$. It means that $\varphi_n \to \varphi^*$ are strongly in $H^1(\Omega)$. Up to now, we complete the proof of the properties (i) and (ii).

The proof of property (iii) as follows. Because $\varphi^* \in H^1(\Omega)$ is a minimizing function, which contents the following Euler-Lagrange equation:

$$\left.\frac{d}{d\theta}\right|_{\theta=0} J_{\alpha,p}(\varphi^* + \theta\zeta) = 0$$

for $\forall\zeta \in C_0^\infty(\Omega)$. On the basis of above argument, and $||\varphi^*||_2 = ||\nabla\varphi^*||_2 = 1$. So we can get

$$\frac{pN-(N+\alpha)}{2}\Delta\varphi^* - \frac{(N+\alpha)-p(N-2)}{2}\varphi^* + \gamma p\int_\Omega \frac{|\varphi^*(y)|^p}{|x-y|^{N-\alpha}}dy|\varphi^*(x)|^{p-2}\varphi^*(x) = 0.$$

Take the scaling $\varphi^* = (\gamma p)^{-\frac{1}{2p-2}}\varphi$, thus φ is a solution of Eq.(11) and $\gamma = \frac{||\varphi||_2^{2(p-1)}}{p}$ because of $||\varphi^*||_2^2 = 1$. This completes the proof. ■

Remark 5 *From* (9) *and* (10)*, we have*

$$C_{N,p} \geq \frac{\left(\int_\Omega |\nabla u|^2\right)^{\frac{pN-(N+\alpha)}{2}}\left(\int_\Omega |u|^2\right)^{\frac{(N+\alpha)-p(N-2)}{2}}}{\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy} = J_{\alpha,p}(u)$$

By the definition of γ *in Lemma* 4 *and above argument, we get that*

$$C_{N,p} = \inf_{u\in H^1(\Omega)} J_{\alpha,p}(u) = \gamma = \frac{||\varphi||_2^{2(p-1)}}{p},$$

where φ *is the ground state solution of the problem* (11)*. And this is the expression of the best constant we want to find.*

5 The existence of solutions about problem (3)

In this section, we find the solutions about problem (3) in the cases of subcritical, critical and supercritical.

Theorem 6 *(subcritical and critical case: existence) Let* $\lambda > 0$, $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha+2}{N}$ *or* $p = \frac{N+\alpha+2}{N}$ *for* $0 < \mu < \mu^*$*. Then problem* (3) *admits at least one global solution.*

Proof. We demonstrate from the following two steps.
Step 1. The energy functional

$$\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2dx - \frac{\mu}{2p}\int_\Omega \phi_u(x)|u|^pdx, \text{ where } \phi_u(x) = \int_\Omega \frac{|u(y)|^p}{|x-y|^{N-\alpha}}dy, \tag{12}$$

is coercive. First, notice that the Gagliardo-Nirenberg inequality:

$$\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \leq S||u||_2^{(N+\alpha)-p(N-2)}||\nabla u||_2^{pN-(N+\alpha)}, \tag{13}$$

where $\alpha \in (0,N)$, $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2} \leq \frac{N+\alpha+2}{N}$, $S > 0$ is a constant. And since $||u||_2 = 1$, we have

$$\begin{aligned}\varepsilon(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2dx - \frac{\mu}{2p}\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \\ &\geq \frac{1}{2}||u||^2_{H_0^1(\Omega)} - \frac{\mu}{2p}S||u||^{pN-(N+\alpha)}_{H_0^1(\Omega)}.\end{aligned}$$

When $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha+2}{N}$ or for $\frac{1}{2} - \frac{\mu}{2p}S > 0$ under the case of $p = \frac{N+\alpha+2}{N}$, namely $0 < \mu < \frac{p}{S} = \mu^*$, we can find that $\varepsilon(u)$ is coercive.
Step 2. We want to prove that the infimum of $\varepsilon(u)$ is achieved.
Let $m := \inf_{||u||_2=1} \varepsilon(u)$. From Step 1, we know that $m > -\infty$, it will be also immediately follow the fact that m is achieved. We elaborate it as follows.

Take a minimizing sequence $\{u_n\}_n \subset H_0^1(\Omega)$, we can easily get that $\{u_n\}_n$ is a bounded sequence in $H_0^1(\Omega)$ by Step 1. Thus, we might as well suppose that u_n is the convergence subsequence, which satify

- $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$;

- $u_n \to u$ strongly in $L^2(\Omega)$;
- $u_n(x) \to u(x)$ almost everywhere in Ω.

Now, let $G(u(x)) := \phi_u(x)|u(x)|^{p-2}u(x)$. Because $G(u(x))$ is continuous, we obtain $G(u_n(x)) \to G(u(x))$ a.e. in Ω. Due to Ω is a bounded domain, we have $G(u_n) \to G(u)$ in $L^1(\Omega)$ from dominated convergence. And there is the asymptotic $\int_\Omega G(u_n)dx \to \int_\Omega G(u)dx$ holds.

From the weakly lower semicontinuity, we have

$$\begin{aligned}\varepsilon(u) &= \frac{1}{2}||u||^2_{H^1_0(\Omega)} - \int_\Omega G(u)dx \leq \lim_n inf \frac{1}{2}||u_n||^2_{H^1_0(\Omega)} - \lim_n \int_\Omega G(u_n)dx \\ &= \lim_n inf \left(\frac{1}{2}||u_n||^2_{H^1_0(\Omega)} - \int_\Omega G(u_n)dx\right) \\ &= \lim_n inf \varepsilon(u_n) = m.\end{aligned}$$

And there is a function $u \in H^1_0(\Omega)$ such that $\varepsilon(u) \geq m$, it means that $\varepsilon(u) = m$. Then, u is a global minimum point about ε. So u also is a critical value point, that is the solution of problem (3). ■

Remark 7 *From Theorem* 6*, we know that the problem* (3) *admits at least one global solution when* $p = \frac{N+\alpha+2}{N}$ *for* $0 < \mu < \mu^* = \frac{p}{S}$*. By Lemma 4, for the best constant* $S = C_{N,p} = \frac{||\varphi||_2^{2(p-1)}}{p}$*, we can write the concrete expression that is* $\mu^* = ||\varphi||_2^{2(p-1)}$*, where* φ *is the ground state solution of the problem* (11)*.*

In Theorem 6, we proved that $\varepsilon(u)$ is coercive under $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha+2}{N}$ or for $0 < \mu < \mu^*$ under $p = \frac{N+\alpha+2}{N}$. In the supercritical case $p > \frac{N+\alpha+2}{N}$, the previous approach cannot work, since $\varepsilon(u)$ restricted to manifold $\mathcal{M} := \{u \in H^1_0(\Omega) : \int_\Omega u^2dx = 1\}$ is not coercive, as we show in the following lemma.

Lemma 8 *(supercritical: anticoercive) Let* $\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$*. Thus, there are a nonnegative sequence* $u_k \in \mathcal{M}$ *such that*

$$||u_k||_{H^1_0(\Omega)} \to +\infty \text{ and } \varepsilon(u_k) \to -\infty, \text{ as } k \to +\infty.$$

Proof. Take $\phi \in C_c^\infty(B_1)$ and $\phi > 0$ in B_1 such that $\int_{B_1}\phi^2 = 1$. For every $k \in \mathbb{N}$, define

$$u_k(x) := k^{\frac{N}{2}}\phi(kx),\ u_k(y) = k^{\frac{N}{2}}\phi(ky),\ \text{for } \forall x, y \in \Omega.$$

Let k large enough, then we can get

$$\int_\Omega u_k^2 = \int_\Omega k^N\phi^2(kx)dx = \int_{B_1}\phi^2 = 1.$$

Hence $u_k \in \mathcal{M}$ for k large enough. We directly calculate

$$||u_k||^2_{H^1_0(\Omega)} = \int_\Omega |\nabla u_k|^2dx = \int_\Omega k^N|\nabla\phi(kx)|^2dx = k^2||\nabla\phi||^2_{L^2(B_1)},$$

and then $||u_k||_{H^1_0(\Omega)} = k||\nabla\phi||_{L^2(B_1)} \to +\infty$ as $k \to +\infty$. In addition,

$$\begin{aligned}\varepsilon(u_k) &= \frac{1}{2}\int_\Omega |\nabla u_k|^2dx - \frac{\mu}{2p}\int_\Omega\int_\Omega \frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}dxdy \\ &= \frac{k^2}{2}||\nabla\phi||^2_{L^2(B_1)} - \frac{\mu}{2p}k^{Np}\int_\Omega\int_\Omega \frac{\phi^p(k(x-x_0))\phi^p(k(y-y_0))}{|x-y|^{N-\alpha}}dxdy \\ &= \frac{k^2}{2}||\nabla\phi||^2_{L^2(B_1)} - \frac{\mu}{2p}k^{N(p-2)+N-\alpha}\int_{B_1}\int_{B_1}\frac{\phi^p(x)\phi^p(y)}{|x-y|^{N-\alpha}}dxdy \to -\infty,\end{aligned}$$

as $k \to +\infty$ because of $2 < N(p-2)+N-\alpha$. This completes the proof of Lemma 8. ■

In supercritical case, we try to find the critical value by Mountain Pass Lemma in the following theorem.

Theorem 9 *(supercritical case: existence) Let* $\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$*, the problem* (3) *has at least one nontrivial solution.*

Proof. Firstly, we can easily prove that the functional $\varepsilon(u)$ is $C^{1,1}(H_0^1(\Omega))$. Secondly, the functional $\varepsilon(u)$ satisfied Palais-Smale condition for all $c \in \mathbb{R}$. We divided it into the following two steps to illustrate.
Step 1. We need show that the Palais-Smale sequence $\{u_n\}$ is bounded.

Let $c \in \mathbb{R}$, suppose $\{u_n\}$ is the Palais-Smale sequence at level c. That is

$$\varepsilon(u_n) \to c \text{ and } \varepsilon^{'}(u_n) \to 0 \text{ with } ||u_n||_2 = 1.$$

Thus, there exists a positive constant C_1, so that

$$|\varepsilon(u_n)| \leq C_1 \text{ and } |\varepsilon^{'}(u_n)u_n| \leq C_1||u_n||_{H_0^1(\Omega)}.$$

So we have

$$\left|\varepsilon(u_n) - \frac{1}{2p}\varepsilon^{'}(u_n)u_n\right| \leq |\varepsilon(u_n)| + |\varepsilon^{'}(u_n)u_n| \leq C_1\left(1 + ||u_n||_{H_0^1(\Omega)}\right), \tag{14}$$

and

$$\left|\varepsilon(u_n) - \frac{1}{2p}\varepsilon^{'}(u_n)u_n\right| = \left|\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \frac{1}{2p}\int_\Omega |\nabla u_n|^2\right| = \frac{p-1}{2p}||u_n||^2_{H_0^1(\Omega)}. \tag{15}$$

Combine (14) with (15), we obtain $||u_n||_{H_0^1(\Omega)}$ is bounded. Namely, $\{u_n\}$ is bounded in $H_0^1(\Omega)$.
Step 2. We are going to show that $\{u_n\}$ is convergence strongly.

Because $\{u_n\}$ is bounded in $H_0^1(\Omega)$ from Step 1, there exists a subsequence $\{u_n\}$, we also denote $\{u_n\}$, satisfy

- $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, where $u \in H_0^1(\Omega)$;
- $u_n \to u$ strongly in $L^q(\Omega)$ for every $q \in (2, 2^*)$.

And $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Now it is easy to conclude that $u_n \to u$ in $H_0^1(\Omega)$, and we have

$$o(1) = \left(\varepsilon^{'}(u_n) - \varepsilon^{'}(u)\right)(u_n - u) = ||u_n - u||^2_{H_0^1(\Omega)} + o(1).$$

By the directly computation, we obtain

$$\left(\varepsilon^{'}(u_n) - \varepsilon^{'}(u)\right)(u_n - u) = ||u_n - u||^2_{H_0^1(\Omega)} - \mu\int_\Omega\int_\Omega \frac{u_n^{p-1}(x)u_n^p(y) - u^{p-1}(x)u^p(y)}{|x-y|^{N-\alpha}}(u_n - u)dxdy.$$

We only need to prove that $\int_\Omega\int_\Omega \frac{u_n^{p-1}(x)u_n^p(y)}{|x-y|^{N-\alpha}}(u_n - u)dxdy = o(1)$ as $n \to \infty$. From the Hardy-Littlewood inequality:

$$\int_\Omega\int_\Omega \frac{u_n^{p-1}(x)(u_n - u)u_n^p(y)}{|x-y|^{N-\alpha}}dxdy \leq C\left[\int_\Omega (u_n - u)^{\frac{2pN}{N+\alpha}}dx\right]^{\frac{N+\alpha}{2pN}}\left[\int_\Omega u_n^{\frac{2pN}{N+\alpha}}(x)dx\right]^{\frac{p(N+\alpha)}{2(p-1)N}}\left[\int_\Omega u_n^{\frac{2pN}{N+\alpha}}(y)dy\right]^{\frac{N+\alpha}{2N}}.$$

Since $\frac{N+\alpha}{N} < \frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$, that is $2 < \frac{2pN}{N+\alpha} < 2^*$. So that we have $u_n \to u$ strongly. Thus the Palais-Smale condition holds for all c.

Finally, we found that the functional $\varepsilon(u)$ has the Mountain Pass geometric structure. According to the definition of Mountain Pass geometry, we prove it from three points. Obviously, $\varepsilon(0) = 0$. And then, we need show that there exists two positive constants a, b, if $||u||_{H_0^1(\Omega)} = a$, such that $\varepsilon(u) = b$.

For all $u \in H_0^1(\Omega)$, from Gagliardo-Nirenberg inequality and $||u||_2 = 1$, we get that

$$\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2p}\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \geq \frac{1}{2}||u||^2_{H_0^1(\Omega)} - \frac{\mu}{2p}C_{N,p}||u||^{2p}_{H_0^1(\Omega)}.$$

Since $p > \frac{N+\alpha+2}{N} > 1$, it is enough for us to find a small positive a and $||u||_{H_0^1(\Omega)} = a$, such that $\frac{1}{2}a^2 - \frac{\mu}{2p}C_{N,p}a^{2p} > 0$.

At last, we only need to show that there exists a function $v \in H_0^1(\Omega)$, such that $||v||_{H_0^1(\Omega)} > a$ and $\varepsilon(v) < b$.

For all $u \in H_0^1(\Omega)\backslash\{0\}$ and $s > 0$, we have

$$\varepsilon(su) = \frac{s^2}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu s^{2p}}{2p}\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \geq \frac{s^2}{2}||u||^2_{H_0^1(\Omega)} - \frac{\mu s^{2p}}{2p}\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy.$$

Since $p > 1$, $\mu > 0$, $||u||^2_{H_0^1(\Omega)} > 0$ and $\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy > 0$, then $\varepsilon(su) \to -\infty$ as $s \to +\infty$. Then we can get that there exists a functin $v \in H_0^1(\Omega)$, such that $||v||_{H_0^1(\Omega)} > a$ and $\varepsilon(v) < 0$.

In summary, by Mountain Pass Lemma, we find the local nontrivial solution for the problem (3). ■

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