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Existence of Nontrivial Radial Solutions for Schrödinger-Possion System Via Spectral Properties

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Abstract: In this paper we study the existence of nontrivial radial solutions of the nonlinear Schrödinger-Possion system. Under suitable assumptions on the nonlinearity and potential, we prove the existence of radial nontrivial solution by using variational methods.

Keywords: Nonlocal systems, Radial solutions, Variational methods.

1 Introduction

In this paper we are concerned with the existence of nontrivial radial solutions of the following equation in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + V(|x|)u = f(|x|, u) + k(|x|)\phi u, \ x \in \mathbb{R}^3, \\ -\Delta \phi = k(|x|)u^2, \ x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where k satisfies $\lim_{x\to\infty} k(x) = 0$.

In the recent years, the following Schrödinger-Poisson system has been an object of interest for many mathematics.

$$\begin{cases} -\Delta u + \lambda V(x)u + k(x)\phi u = f(x,u) \quad \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2 \quad \text{in} \quad \mathbb{R}^3, \end{cases} \tag{2}$$

where $\lambda \geq 1$ is a parameter. In present paper, we shall concerned with the situation of $\lambda = 1$. This system has been first introduced in [5] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. The unknowns u and ϕ represent the wave functions associated to the particle and electric potential, and functions V and k are respectively an external potential and nonnegative density charge. In fact, the system also arises in semiconductor theory, nonlinear optics and plasma physics. For more details physical background, we refer the readers to [5, 10, 11].

Up till now, under various assumptions on the potential $V(x)$ and the nonlinearity $f(x,u)$, there has been a large number of papers on the study of existence and multiplicity of solutions of system (2) by taking advantages of variational methods. One can refer the papers [1, 2, 4, 5, 10, 13, 25] and the references therein. Most of them solve the problem where V is a positive constant or being radially symmetric and $f(x,u) = |u|^{p-1}u, 1 < p < 5$. In [28], Sánchez and Soler deals with system (2) when $p = \frac{5}{3}, \lambda = 1$. They have obtained the exisitence of a positive solution by minimization on Nehari manifold. Using the mountain pass theorem, the papers [9, 10] proved that system (2) has a radial positive solution for $3 \leq p < 5$. The papers [26] also investigated the existence of multiple solutions for system (2), depending on λ and p.

In [25], Ruiz studied the following system:

$$\begin{cases} -\Delta u + u + \lambda \phi u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = u^2 \quad \text{in} \quad \mathbb{R}^3, \end{cases} \tag{3}$$

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The author shows that if $\lambda \geq \frac{1}{4}$, system (3) does not admit any non-trivial solution for $p \in (1, 2]$, and if $\lambda > 0$ is sufficiently small, it possesses two positive radial solutions for $p \in (1, 2)$ and one positive radial solution for $p = 2$. Moreover, if $p \in (2, 5)$, there exists a positive radial solution for all $\lambda > 0$. Ambrosetti and Ruiz [1] and Ambrosetti [2] also studied problem (3) with a parameter. They constructed the existence of infinitely many pairs of radial solutions, when $2 < p < 5$, for all $\lambda > 0$, and also multiple solutions (but not infinitely many), when $1 < p \leq 2$, for all $\lambda > 0$ small sufficiently.

Very recently, Cerami and Vaira [8] considered the following system:

$$\begin{cases} -\Delta u + u + k(x)\phi u = a(x)|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2 \quad \text{in} \quad \mathbb{R}^3, \end{cases} \tag{4}$$

where $3 < p < 5$, and $k \in L^2(\mathbb{R}^3)$. Applying the Nehari variational principle and concentration compactness argument, they proved that (4) possesses a positive ground state solution when $a : \mathbb{R}^3 \to \mathbb{R}$ and $k : \mathbb{R}^3 \to \mathbb{R}$ are nonnegative functions such that

$$\lim_{|x|\to\infty} a(x) = a_\infty \quad \text{and} \quad \lim_{|x|\to\infty} k(x) = 0.$$

Furthermore, the semiclassical case of system (2) has received great attention in recent years and has been extensively studied using variational methods. Replacing $-\Delta$ by $-\varepsilon\Delta$, D'Aprile and Wei [12] constructed positive radially symmetric bound states of (2) with $f(x, u) = u^p$, $1 < p < \frac{11}{7}$. Recently, making use of a standard Lyapunov-Schmidt reduction methods (see [3]), Ruiz and Vaira [27] proved the existence of multi-bump solutions of (2), whose bumps concentrate around a local minimum of the potential $V(x)$ when $f(x, u) = u^p$ and $3 < p < 5$. The proofs explored in [17, 18] are based on a singular perturbation. In this framework, one is interested not only in existence of solutions but also in their asymptotic behavior as $\varepsilon \to 0$. For more general information on semiclassical states for this system, see for example [2, 15, 17–19, 26, 31–33] and the references therein.

Motivated by the papers [22, 34, 35], in this paper we shall concentrate on the existence of radial nontrivial solution of (2) when $V(x)$ is a sign-changing function. We shall recover the compactness by using special properties of radially symmetric functions. Precisely, we use spectral properties of operator $A = \Delta{+}V(x)$ restricted to $H^1_{rad}(\mathbb{R}^3)$ for obtaining a linking geometry structure to the problem.

In particular, supposing that $V(x)$ satisfies:

(V_1) $V \in L^\infty(\mathbb{R}^3)$ is a radial sign-changing function, $V(x) = V(|x|) = V(r), r \geq 0$.

(V_2) Setting $\tilde{V}(r) = V(r) + V_N$, where $V_N = \frac{(N-1)(N-3)}{4r^2}$ and $\tilde{A} = -\frac{d^2}{dr^2} + \tilde{V}(r)$, an operator of $L^2(0, \infty)$, $0 \notin \sigma_{ess}(\tilde{A})$ and

$$\sup[\sigma(\tilde{A}) \cap (-\infty, 0)] = \sigma^- < 0 < \sigma^+ = \inf[\sigma(\tilde{A}) \cap (0, +\infty)].$$

Moreover, we can assume that the nonlinear function $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfying the following hypotheses:

(f_1) For all $t \in \mathbb{R}$, $F(x, t) = \int_0^t f(x, s)ds \geq 0$ and f is a radial function such that $\lim\limits_{|s|\to 0} \frac{f(x,s)}{s} = 0$, uniformly in x.

(f_2) $\lim\limits_{|s|\to+\infty} \frac{f(x,s)}{s} = h(x)$, uniformly in x, where $h \in L^\infty(\mathbb{R}^3)$.

(f_3) $a_0 = \inf\limits_{x\in\mathbb{R}^3} h(x) > \sigma^+ = \inf[\sigma(A) \cap (0, +\infty)]$.

(f_4) Setting $\mathcal{J} = A - \mathcal{K}$, where $\mathcal{K}$ is the operator multiplication by $(h(x) + k(x)\phi_u)$ in $L^2(\mathbb{R}^3)$ and defining by $\sigma_p(\mathcal{J})$ the point spectrum of $\mathcal{J}$, $0 \notin \sigma_p(\mathcal{J})$.

(f_5) Denoting $T(x, s) = \frac{1}{2}f(x, s)s - F(x, s) \geq 0$ for all $(x, s) \in (\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and exists $\kappa_0 > 0$ such that $\frac{f(x,s)}{s} \geq \kappa_0 \Rightarrow T(x, s) \geq \kappa_0$.

Then the following main results hold.

Theorem 1 *Under the assumptions* (V_1)-(V_2) *and* (f_1)-(f_5)*, the problem* (1) *possess a radial, nontrivial, weak solution in* $H^1(\mathbb{R}^3)$.

We shall use the variational methods to prove main results. The rest of the paper is structured as follows. In next section we first establish the variational setting for the system (1) and then we state our main result. In section 3 the required compactness for the related functional is presented. Section 4 describes how to establish the linking geometry by means of the sharp construction of the linking components based on the spectral results. The core of our arguments is to by means of the strict inequality in (f_3) throughout this section. Finally, we obtained the boundedness of Cerami sequences for the functional and proved the main result.

2 The variational setting

Since $V \in L^\infty(\mathbb{R}^3)$, A and $\tilde{A}$ are self-adjoint operators, we can consider $A = -\Delta + V(x)$ as an operator of $L^2(\mathbb{R}^3)$. From the Hardy's inequality, we know that the operator $\tilde{A}$ is treated in $H_0^1(0,\infty)$. Since H^-, H^0, H^+ are the subspaces of $H_0^1(0,\infty)$ on which $\tilde{A}$ is negative, null and positive definite, it follows that $H_0^1(0,\infty) = H^- \oplus H^0 \oplus H^+$. In the light of (V_2) each $u \in H^+$ satisfies

$$\sigma^+ ||u||^2_{L^2(0,\infty)} \le (\tilde{A}u, u)_{L^2(0,\infty)}.$$

In addition, for $u \in H_0^1(0,\infty)$ and setting $\varphi = r^{\frac{1-N}{2}} u$, it yields $\varphi \in H^1_{rad}(\mathbb{R}^3)$ (see [30, Section 3]), where $H^1_{rad}(\mathbb{R}^3)$ is the Hilbert subspace of all radial symmetric functions in $H^1(\mathbb{R}^3)$. Moreover, we change the variables, then φ satisfies

$$\begin{aligned} ||\varphi||^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\varphi(x)|^2 dx = \varphi_N \int_0^\infty |\varphi(r)|^2 r^{N-1} dr \\ &= \varphi_N \int_0^\infty r^{1-N} |u(r)|^2 r^{N-1} dr = \varphi_N \int_0^\infty |u(r)|^2 dr = \varphi_N ||u||^2_{L^2(0,\infty)}, \end{aligned}$$

and

$$\begin{aligned} (A\varphi, \varphi)_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} (|\nabla \varphi(x)|^2 + V(x)\varphi(x)^2) dx \\ &= \varphi_N \int_0^\infty r^{N-1} \varphi'(r)^2 dr + \varphi_N \int_0^\infty V(x) r^{N-1} \varphi(r)^2 dr \\ &= \varphi_N \int_0^\infty r^{N-1} \left[\frac{1-N}{2} r^{\frac{1-N}{2}-1} u(r) + r^{\frac{1-N}{2}} u'(r) \right]^2 dr + \varphi_N \int_0^\infty V(x) u(r)^2 dr \\ &= \varphi_N \int_0^\infty \left[u'(r)^2 + \frac{(N-1)(N-3)}{4r^2} u(r)^2 \right] dr + \varphi_N \int_0^\infty V(x) u(r)^2 dr \\ &= \varphi_N \int_0^\infty \left[|u'(r)|^2 + V_N u(r)^2 \right] dr + \varphi_N \int_0^\infty V(x) u(r)^2 dr \\ &= \varphi_N \int_0^\infty \left(|u'(r)|^2 + \tilde{V}(r) u^2(r) \right) dr \\ &= \varphi_N (\tilde{A}u, u)_{L^2(0,\infty)}, \end{aligned}$$

where φ_N is the $(N-1)$-dimensional surface measure of the sphere $S^{N-1} \subset \mathbb{R}^N$. Therefore, we have $\sigma^+ ||\varphi||^2_{L^2(\mathbb{R}^3)} \le (A\varphi, \varphi)_{L^2(\mathbb{R}^3)}$. If some function $\tilde{\varphi} \in H^1_{rad}(\mathbb{R}^3)$ satisfies the following inequality

$$0 < (A\tilde{\varphi}, \tilde{\varphi})_{L^2(\mathbb{R}^3)} < \sigma^+ ||\tilde{\varphi}||^2_{L^2(\mathbb{R}^3)},$$

by approximation it can be seen as a smooth function and then setting $\tilde{u} = r^{\frac{N-1}{2}} \tilde{\varphi} \in H^+$ and satisfies

$$\sigma^+ ||\tilde{u}||^2_{L^2(0,\infty)} > (\tilde{A}\tilde{u}, \tilde{u})_{L^2(0,\infty)},$$

which contradicts (V_2). Thus, writing $H^1_{rad}(\mathbb{R}^3) = E^- \oplus E^0 \oplus E^+$, with E^-, E^0, E^+ the subspaces where of A is respectively negative, null and positive definite. If $\varphi \in E^+$ it satisfies $\sigma^+ ||\varphi||^2_{L^2(\mathbb{R}^3)} \le (A\varphi, \varphi)_{L^2(\mathbb{R}^3)}$.

Remark 2 *Note that if σ^+ is an eigenvalue of $\tilde{A}$ with eigenfunction u, the same argument as above shows that σ^+ is an eigenvalue of A, with a radial eigenfunction $\varphi = r^{\frac{1-N}{2}} u \in E^+$. On the other hand, if σ^+ is not an eigenvalue of $\tilde{A}$, since it belongs to $\sigma_{ess}(\tilde{A})$, given $\varepsilon > 0$ there exist $u_\varepsilon \in H^+$ such that*

$$\sigma^+ ||u_\varepsilon||^2_{L^2(0,\infty)} < (\tilde{A}u_\varepsilon, u_\varepsilon)_{L^2(0,\infty)} < (\sigma^+ + \varepsilon)||u_\varepsilon||^2_{L^2(0,\infty)},$$

which ensures that $\varphi_\varepsilon = r^{\frac{1-N}{2}} u_\varepsilon \in E^+$ satisfies

$$\sigma^+ ||\varphi_\varepsilon||^2_{L^2(\mathbb{R}^3)} < (A\varphi_\varepsilon, \varphi_\varepsilon)_{L^2(\mathbb{R}^3)} < (\sigma^+ + \varepsilon)||\varphi_\varepsilon||^2_{L^2(\mathbb{R}^3)}.$$

Thus,

$$\sigma^+ = \inf_{\varphi \in E^+} \frac{(A\varphi, \varphi)_{L^2(\mathbb{R}^3)}}{||\varphi||^2_{L^2(\mathbb{R}^3)}}. \tag{1}$$

Applying the same arguments comparing H^- and E^-, it holds

$$-\sigma^- = \inf_{\varphi \in E^-} \frac{-(A\varphi, \varphi)_{L^2(\mathbb{R}^3)}}{||\varphi||^2_{L^2(\mathbb{R}^3)}}. \tag{2}$$

According to the suppose (V_2), it is easy to know that either $0 \notin \sigma(\tilde{A})$ or it is an isolated eigenvalue of $\tilde{A}$. Since $0 \notin \sigma_{ess}(\tilde{A})$ is assumed, if $0 \in \sigma(\tilde{A})$ it is an eigenvalue of finite multiplicity, hence ker$(\tilde{A})$ is finite dimensional. Because there exists a correspondence between the eigenfunctions of $\tilde{A}$ and the radial eigenfunctions of A, the same results applies to A. Furthermore, $u_1, u_2 \in H^1_0(0,\infty)$ are orthogonal in $L^2(0,\infty)$ iff $\varphi_1 = r^{\frac{1-N}{2}} u_1$ and $\varphi_2 = r^{\frac{1-N}{2}} u_2$ are orthogonal in $L^2(\mathbb{R}^3)$. In fact,

$$\int_0^\infty u_1(r)u_2(r)dr = \frac{1}{\varphi_N} \int_{\mathbb{R}^3} \varphi_1(x)\varphi_2(x)dx.$$

Hence, H^i is infinite dimensional iff E^i is infinite dimensional, for $i = -, 0, +$.

A typical example of V satisfying $(V_1) - (V_2)$ is a suitable continuous, periodic and sign-changing $V(r)$, such that $0 \notin \sigma\left(-\frac{d^2}{dr^2} + V(r)\right)$. Therefore 0 is in the gap of the spectrum, which is composed of closed intervals. $(-\frac{d^2}{dr^2} + V(r)$ has positive and negative spectrum, since $V(r)$ is continuous and sign-changing. In addition, $\tilde{V} = V + V_N$, where $V_N(r)$ decays fast enough, then it is a Kato's potential and hence $\tilde{A}$-compact, which ensures $\sigma_{ess}(\tilde{A}) = \sigma\left(-\frac{d^2}{dr^2} + V(r)\right)$ by Weyl's theorem (see[23, Corollary 11.3.6] or [16, sections 14.2-14.3]), hence $0 \notin \sigma_{ess}(\tilde{A})$ and $\sigma(\tilde{A})$ also has positive and negative part. Therefore, (V_2) is satisfied.

Remark 3 *Examples of potentials which satisfy or not assumptions:*

(E_1) *On the basis of the previous observations, $V(r) = cos(r)$ satisfies $(V_1) - (V_2)$.*

(E_2) *If $V(r) = \frac{1}{1+r^2} - \frac{1}{2}$ it not satisfies (V_2) since $0 \in \sigma_{ess}(\tilde{A})$. In fact, $\lim_{r\to+\infty} V(r) = -\frac{1}{2}$, then $\sigma_{ess}(\tilde{A}) = \sigma_{ess}(A) = [-\frac{1}{2}, +\infty)$.*

Next we give an example of f satisfying $(f_1) - (f_5)$, which is an asymptotically linear continuous function such that $h(x) \equiv a_0 > \sigma^+$ as in (f_3), then for a periodic V, since $\sigma(A)$ is pure absolutely continuous, $a_0 \notin \sigma_p(A)$ and hence $0 \notin \sigma_p(\mathcal{J})$ as in (f_4). For example,

$$f(x,s) = \frac{s^3}{1 + a_0^{-1}s^2} \quad \text{and} \quad f(x,s) = (a_0 - \frac{1}{\exp s^2})s.$$

Remark 4 *By $(f_1) - (f_2)$, given $\varepsilon > 0$ and $2 \le p \le 2^*$ there exists a constant $C_\varepsilon > 0$ for all $s \in \mathbb{R}$ and for all $x \in \mathbb{R}^3$ such that*

$$|f(x,s)| \le \varepsilon |s| + C_\varepsilon |s|^{p-1}, \tag{3}$$

and hence

$$|F(x,s)| \le \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{p}|s|^p. \tag{4}$$

For any given $u \in E$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is a weak solution to the following Possion equation

$$-\Delta\phi = k(x)u^2 \quad x \in \mathbb{R}^3, \tag{5}$$

and define $\Phi : E \to D^{1,2}(\mathbb{R}^3)$, it can be expressed explicitly as (see [11])

$$\Phi(u) = \phi_u = \int_{\mathbb{R}^3} \frac{k(y)u^2(y)}{|x-y|} dy.$$

Next we give some properties about Φ as followings (cf.[7, 8]).

Lemma 5 *The following results hold.*

(1) The functional Φ is continous.

(2) Φ maps bounded sets into bounded sets.

(3) $\Phi[tu] = t^2\Phi[u]$ for all $t \in \mathbb{R}$.

(4) For $u \in H^1(\mathbb{R}^3)$, if $u_n \rightharpoonup u$, then $\Phi[u_n] \to \Phi[u]$ in $D^{1,2}(\mathbb{R}^3)$. Moreover, $\int_{\mathbb{R}^3} k(x)\phi_{u_n}(x)u_n^2 \to \int_{\mathbb{R}^3} k(x)\phi_u(x)u^2$ and $\int_{\mathbb{R}^3} k(x)\phi_{u_n}(x)u_n\varphi \to \int_{\mathbb{R}^3} k(x)\phi_u(x)u\varphi, \forall\varphi \in H^1(\mathbb{R}^3)$.

Proof. The proof of the conclusions (1) and (2) can be found in [8]. Properties (3) is direct consequences of the definition of ϕ_u as a weak solution of (5). Finally, the proof of (4) given in [7] can be applied. We omit the details in this paper. ■

From the above computations, the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated to problem (1)

$$I(u) = \frac{1}{2}(Au, u)_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} F(x, u)dx - \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx \tag{6}$$

is well defined. Furthermore, it is known that I is a C^1 functional with derivative given by

$$I'(u)v = (Au, v)_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} f(x, u(x))v(x)dx - \int_{\mathbb{R}^3} k(x)\phi_u uvdx = 0.$$

Therefore, a weak solution for (2) is a critical point of $I : H^1(\mathbb{R}^3) \to \mathbb{R}$.

To obtain a nontrivial critical point of the functional I, we used the abstract linking theorem (see [21]), now we recall the details below.

Theorem 6 *(Linking theorem for Cerami sequences). Let E be a real Hilbert space, with inner product $(\cdot, \cdot)$, E_1 a closed subspace of E and $E_2 = E_1{}^{\perp}$. Let $I \in C^1(E, \mathbb{R})$ satisfying:*

(I_1) *$I(u) = \frac{1}{2}(Lu, u) + B(u)$, for all $u \in E$, where $u = u_1 + u_2 \in E_1 \oplus E_2$, $Lu = L_1u_1 + L_2u_2$ and $L_i : E_i \to E_i, i = 1, 2$ is a bounded linear self-adjoint mapping.*

(I_2) *B is weakly continuous and uniformly differentiable on bounded subsets of E.*

(I_3) *There exist Hilbert manifolds $S, Q \subset E$, such that Q is bounded and has boundary ∂Q, constants $\alpha > \omega$ and $v \in E_2$ such that*

(i) $S \subset v + E_1$ and $I \geq \alpha$ on S; (ii) $I \leq \omega$ on ∂Q; (iii) S and ∂Q link.

(I_4) *If for a sequence (u_n), $I(u_n)$ is bounded and $(1 + \|u_n\|)\,\|I'(u_n)\| \to 0$, as $n \to +\infty$, then (u_n) is bounded.*

Then I possesses a critical value $c \geq \alpha$.

In the following we just need to find a critical point of I by applying Theorem 6. Since I is an indefinite functional, it is necessary to check that I satisfies (I_1)-(I_4). Since V and F are radial functions, for reader's convenience, we define $E = H^1_{rad}(\mathbb{R}^3)$, which is the Hilbert subspace of all radial symmetric functions in $H^1(\mathbb{R}^3)$ and consider $I : E \to \mathbb{R}$. In fact, functions in E satisfy special properties that make true all necessary assumptions on $I : E \to \mathbb{R}$, for instance, for any $\beta \in (2, 2^*)$, E is compactly embedded in $L^\beta(\mathbb{R}^3)$ (see [29] or [6, Theorem A.I']).

We denote $P_A : E \to \mathbb{R}$ is a continuous quadratic form on E and

$$P_A(u) = \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^3} V(x)u^2(x)dx = \frac{1}{2}(Au, u)_{L^2(\mathbb{R}^3)}.$$

The space E can be written as $E = E^0 \oplus E^- \oplus E^+$ with E^0, E^-, E^+ are the closed subspaces of E where P_A is null, negative and positive definite. Moreover, for all $u, v \in E$, if $B_{P_A}[u, v] = (Au, v)_{L^2(\mathbb{R}^3)}$ is the bilinear form associated to P_A and u, v belong to distinct such subspaces, then one has $B_{P_A}[u, v] = 0$ and $P_A(u+v) = P_A(u) + P_A(v)$. In addition E^-, E^0, E^+ are mutually orthogonal in the $L^2(\mathbb{R}^3)$-inner product. Thus, for $u = u^0 + u^- + u^+ \in E$, it is suitable to take as an equivalent norm in E the expression

$$\|u\|^2 = \|u\|^2_E = \left\|u^2\right\|^2_2 + P_A(u^+) - P_A(u^-),$$

and the associated inner product, obtained through $B_{P_A}[u, v]$, which makes E a Hilbert space with E^-, E^0, E^+ orthogonal subspaces of E. In virtue of (V_2) and Remark 2 for all $u^+ \in E^+$ and for all $u^- \in E^-$, it is possible to yields that

$$\sigma^+ \left\|u^+\right\|^2_2 \leq \int_{\mathbb{R}^3} (\left|\nabla u^+(x)\right|^2 + V(x)(u^+(x))^2)dx = \left\|u^+\right\|^2, \tag{7}$$

and

$$-\sigma^- \left\|u^-\right\|^2_2 \leq -\int_{\mathbb{R}^3} (\left|\nabla u^-(x)\right|^2 + V(x)(u^-(x))^2)dx = \left\|u^-\right\|^2, \tag{8}$$

which ensures that the norm chosen above is equivalent to the standard norm in $H^1_{rad}(\mathbb{R}^3)$, once $E^0 = \ker(A)$ is finite dimensional. It is observed that $I(u) = P_A(u) - \int_{\mathbb{R}^3} F(x, u(x))dx - \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx$, for all $u \in E$ and since E is a subspace of $H^1(\mathbb{R}^3)$, $I \in C^1(E, \mathbb{R})$. In addition, I is indefinite on E, henceforth the goal is to apply Theorem 6 in order to get a critical point of I restricted to E, and by means of the Principle of Symmetric Critically conclude the critical point is actually a critical point of $I : H^1(\mathbb{R}^3) \to \mathbb{R}$, namely a weak solution to (1)(see[24]).

In order to prove that I satisfies (I_1) in Theorem 6. We set $E_1 = E^+$ and $E_2 = E^- \oplus E^0$. Then $E_2^\perp = E_1$ holds. Now, define $L_i : E_i \to E_i$, for all $u \in E_i$, as given by

$$(L_i u, v)_E = P'_A(u)v = B_{P_A}[u, v] = (Au, v)_{L^2(\mathbb{R}^3)},$$

for all $v \in E_i$, $i = 1, 2$, where $P'_A(u)v$ denotes Fréchet derivative of P_A at u acting on v. Thus, $L = L_1 + L_2 : E_1 \oplus E_2 \to E_1 \oplus E_2$ is a well defined, linear, bounded operator and it follows that

$$P_A(u) = \frac{1}{2}(Au, u)_{L^2(\mathbb{R}^3)} = \frac{1}{2}P'_A(u)u = \frac{1}{2}B_{P_A}[u, u] = \frac{1}{2}(Lu, u)_E.$$

Therefore, I satisfies (I_1) in Theorem 6 since

$$I(u) = \frac{1}{2}(Lu, u) + B(u),$$

where

$$B(u) = -\int_{\mathbb{R}^3} F(x, u(x))dx - \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx = B_1(u) + B_2(u) \quad \text{for all } u \in E.$$

3 The weak continuity and uniform differentiation of I

In order to proceed with the proof of I satisfies (I_2), we quote the following famous Strauss compactness lemma[29] (also [6, Theorem A.I.]). The version applies to functions P depending also on the space variable x. Supposing that the dependence is uniform on x as $|s|$ goes to 0 and ∞. With a minor adaptation of the proof given in [29] one can state the following version.

Lemma 7 *Let $P:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ and $Q:\mathbb{R}\to\mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(x,s)}{Q(s)}\to 0,\quad \text{uniformly in x as } |s|\to+\infty. \tag{9}$$

Let (u_n) be a sequence of measurable functions from $\mathbb{R}^3$ to $\mathbb{R}$ such that

$$\sup_n\int\limits_{\mathbb{R}^3}|Q(u_n(x))|dx<+\infty, \tag{10}$$

and

$$P(x,u_n(x))\to v(x)\ a.e.\ in\ \mathbb{R}^3,\ as\ n\to+\infty. \tag{11}$$

Then for any bounded Borel set $\mathcal{B}$ one has

$$\int\limits_{\mathcal{B}}|P(x,u_n(x))-v(x)|\,dx\to 0,\ as\ n\to+\infty. \tag{12}$$

If one further assumes that

$$\frac{P(x,s)}{Q(s)}\to 0,\quad \text{uniformly in x as } s\to 0, \tag{13}$$

and

$$u_n(x)\to 0\ as\ |x|\to+\infty,\ \text{uniformly with respect to n}, \tag{14}$$

then $P(\cdot,u_n(\cdot))$ converges to v in $L^1(\mathbb{R}^3)$ as $n\to+\infty$.

Proof. For the purpose of prove the first part of the proposition, it is sufficient to show that $P(x,u_n(x))$ is uniformly integrable on $\mathcal{B}$. As a matter of fact, in virtue of (11)

$$\int\limits_{\mathcal{B}\cap\{|P(x,u_n(x))|\le r\}}|P(x,u_n(x))-v(x)|\,dx\to 0\quad \text{as } n\to+\infty,$$

by applying Lebesgue Dominated Convergence Theorem. Moreover, the integral

$$\int\limits_{\mathcal{B}\cap\{|P(x,u_n(x))|>r\}}|P(x,u_n(x))|\,dx,$$

is controlled by uniform integration. By condition (9) there exists a positive constant C such that

$$|P(x,u_n(x))|\le C(1+|Q(u_n(x))|),\quad x\in\mathbb{R}^3.$$

Hence, due to (10) and Fatou's Lemma, we can infer that $P(\cdot,u_n(\cdot))$ and v are in $L^1(\mathcal{B})$. In addition, since P is continuous, it maps compact sets on compact sets, thus fixed $r>0$, if for some $x\in\mathbb{R}$, $|P(x,u_n(x))|>r$, there exists $N=N(r)>0$, such that $|u_n(x)|>N(r)$ and $N(r)\to+\infty$ as $r\to+\infty$. Then one has

$$\int\limits_{\mathcal{B}\cap\{|P(x,u_n(x))|>r\}}|P(x,u_n(x))|\,dx\le\int\limits_{\mathcal{B}\cap\{|u_n(x)|>N(r)\}}|P(x,u_n(x))|\,dx.$$

Using condition (9), given $\varepsilon > 0$ there exist $N(r) > 0$, such that $|u_n(x)| \geq N(r)$ implies $|P(x, u_n(x))| \leq \varepsilon\,|Q(u_n(x))|$ and $\varepsilon = \varepsilon(r) \to 0$ as $N(r) \to +\infty$. Then, there exist $\bar{C} > 0$ such that

$$\begin{aligned}\int\limits_{\mathcal{B}\cap\{|P(x,u_n(x))|>r\}} |P(x,u_n(x))|\,dx &\leq \int\limits_{\mathcal{B}\cap\{|u_n(x)|>N(r)\}} |P(x,u_n(x))|\,dx \\ &\leq \varepsilon(r)\int\limits_{\mathcal{B}} |Q(u_n(x))|\,dx \\ &\leq \bar{C}\varepsilon(r),\end{aligned}$$

which implies that the uniform integrability and ensures the claim.

For the second part, $P(\cdot, u_n(\cdot))$ converges to v in $L^1(\mathbb{R}^3)$ as $n \to +\infty$, it is observe that from (13) given $\varepsilon > 0$ there exists $\delta > 0$ such that $|s| \leq \delta$ shows $|P(x, s)| \leq \varepsilon|Q(s)|$, uniformly in x. Moreover, as result of (14) given $\delta > 0$ there exists $r_0 > 0$ such that $|u_n(x)| \leq \delta$ for all $|x| \geq r_0$, uniformly in n. Therefore, $|x| \geq r_0$ implies $|P(x, u_n(x))| \leq \varepsilon|Q(u_n(x))|$, uniformly in n. Hence, in view of Fatou's Lemma $v \in L^1(\mathbb{R}^3)$ and

$$\int\limits_{\{|x|\geq r_0\}} |v(x)|\,dx \leq \liminf_{n\to\infty} \int\limits_{\{|x|\geq r_0\}} |P(x,u_n(x))|\,dx \ \leq \bar{C}\varepsilon.$$

Furthermore, from the first part, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\int\limits_{\{|x|<r_0\}} |P(x,u_n(x)) - v(x)|\,dx \ \leq \varepsilon.$$

Hence, for $n \geq n_0$ we have

$$\int\limits_{\mathbb{R}^3} |P(x,u_n(x)) - v(x)|\,dx \ \leq (2\bar{C}+1)\varepsilon,$$

which gives the result. ■

With the purpose of proving that I satisfies (I_2) in Theorem 6, and by means of the previous lemma, the following results are stated and proved.

Lemma 8 *If f satisfies (f_1)-(f_2), then B is weakly continuous.*

Proof. Let $(u_n) \in E$ and suppose $(u_n) \rightharpoonup u$ in E, then (u_n) is bounded in E. On the one hand, since (f_1)-(f_2), for $2 < p < 2^*$, we get that

$$\lim_{s\to 0} \frac{F(x,s)}{|s|^2} = 0 \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{F(x,s)}{|s|^p} = 0, \ \text{uniformly in } x. \tag{15}$$

Hence, setting $Q(s) = |s|^2 + |s|^p$, and $P(\cdot, s) = F(\cdot, s)$, it is possible to apply Lemma 7. Obviously, in virtue of (15) it yields that

$$\lim_{s\to 0} \frac{F(x,s)}{|s|^2+|s|^p} = 0 \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{F(x,s)}{|s|^2+|s|^p} = 0, \ \text{uniformly in } x. \tag{16}$$

Then P and Q satisfy (9) and (13). Moreover, since (u_n) is bounded in E and E is continuously embedded in $L^2(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$, one has

$$\sup_n \int\limits_{\mathbb{R}^3} (|u_n(x)|^2 + |u_n(x)|^p)dx = \sup_n (\|u_n\|_2^2 + \|u_n\|_p^p) \leq C < +\infty. \tag{17}$$

Hence (10) is satisfied. Provided that $u_n \rightharpoonup u$ in E and E is compactly embedded in $L^p(\mathbb{R}^3)$, $u_n \to u$ in $L^p(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^3$. Thus, letting $v(x) = F(x, u(x))$ it follows that (11) is satisfied. Finally, since $(u_n) \subset H^1_{rad}(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^3$, it yields $\lim\limits_{|x|\to+\infty} u_n(x) = 0$, uniformly with respect to n

(see [6, Lemma A.II.]). Therefore, by Lemma 7, we see that $F(\cdot, u_n(\cdot)) = P(\cdot, u_n(\cdot)) \to v = F(\cdot, u(\cdot))$ in $L^1(\mathbb{R}^3)$ as $n \to +\infty$. In other words,

$$B_1(u_n) = -\int\limits_{\mathbb{R}^3} F(x, u_n(x))dx \to -\int\limits_{\mathbb{R}^3} F(x, u(x))dx = B_1(u), \text{ as } n \to +\infty.$$

On the other hand, applying Lemma 5, we infer that

$$B_2(u_n) = -\frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_{u_n}(x)u_n^2 \to -\frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_u(x)u^2 = B_2(u), \text{ as } n \to +\infty.$$

Then B is weakly continuous and we complete the proof of Lemma 8. ■

In order to prove B is uniformly differentiable on bounded sets of E, we stated the classical Hardy-Littlewood-Sobolev inequality (see [20]), there exists an absolute constant C_0 such that

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy \le C_0\left(\int\limits_{\mathbb{R}^3} |u|^{\frac{12}{5}}\right)^{\frac{5}{3}}. \tag{18}$$

Moreover, we will always use the above inequality in the rest of this paper.

Lemma 9 *Assume that $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$. Then one has*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|}dxdy \le c|f|_p|g|_q, \tag{19}$$

where $1 < p$, $q < \infty$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{3} = 2$.

The following basic inequality is of fundamental importance for considering (1). The proof is provided in [14, Proof of Propostion 3.2 (3.3)].

Lemma 10 *Suppose that $u, v \in H^1(\mathbb{R}^3)$. Then one has*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|v(y)|^2}{|x-y|}dxdy \le \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^2|v(y)|^2}{|x-y|}dxdy\right)^{\frac{1}{2}}. \tag{20}$$

In the following, to simplify our notation, we define

$$\phi_f = \int_{\mathbb{R}^N} \frac{k(y)|f(y)|^2}{|x-y|}dy, \quad f \in H^1_{rad}(\mathbb{R}^3). \tag{21}$$

Apparently, we have the following symmetry property of ϕ_f for $u, v \in H^1_{rad}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} k(x)\phi_u v^2(x)dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k(x)k(y)\frac{u^2(x)v^2(y)}{|x-y|}dxdy = \int_{\mathbb{R}^3} k(y)\phi_v u^2(x)dx, \tag{22}$$

$$\int_{\mathbb{R}^3} k(x)\phi_u u^2(x)dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k(x)k(y)\frac{u^2(x)u^2(y)}{|x-y|}dxdy = \int_{\mathbb{R}^3} k(y)\phi_v v^2(x)dx, \tag{23}$$

$$\int_{\mathbb{R}^3} k(x)\phi_u u(x)v(x)dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k(x)k(y)\frac{u(x)v(x)v^2(y)}{|x-y|}dxdy = \int_{\mathbb{R}^3} k(y)\phi_v v(x)u(x)dx, \tag{24}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k(x)k(y)\frac{u(y)v(y)u^2(x)}{|x-y|}dxdy = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} k(x)k(y)\frac{u(y)v(y)v^2(x)}{|x-y|}dxdy. \tag{25}$$

Lemma 11 *Suppose that f satisfies (f_1)-(f_2), then B is uniformly differentiable on bounded sets of E.*

Proof. First, fixed $\varrho > 0$ and given $u+v, v \in B_\varrho \subset E$, the closed ball centered on the origin, let $\gamma(x) := |f(x,\ \omega(x)) - f(x,\ u(x))|$ and $\omega(x) = u(x) + \theta(x)v(x)$, with $0 \leq \theta(x) \leq 1$ given by Mean Value Theorem and $C_1 > 0$ is the constant which is given by the continuous embedding $E \hookrightarrow L^2(\mathbb{R}^3)$, it is easy to see that

$$\begin{aligned} &\left|B_1(u+v) - B_1(u) - {B_1}'(u)v\right| \\ =&\int\limits_{\mathbb{R}^3} |F(x,u(x)+v(x)) - F(x,u(x)) - f(x,u(x))v(x)|\,dx \\ \leq&\int\limits_{\mathbb{R}^3} |f(x,\omega(x)) - f(x,u(x))|\,|v(x)|\,dx \\ \leq& C_1\|\gamma\|_{L^2(\mathbb{R}^3)}\,\|v\|\,. \end{aligned} \tag{26}$$

To prove that B_1 is uniformly differentiable on bounded sets of E, given $\varepsilon > 0$ it is sufficient to show there exist $\delta > 0$ such that $C_1||\gamma||_{L^2(\mathbb{R}^3)} \leq \varepsilon$ for all $u+v, v \in B_\varrho$ with $\|v\| \leq \delta$. Seeking a contradiction, assume that it is not the case, then there exist $\varrho_0, \varepsilon_0 > 0$ such that for all $\delta > 0$ there are $u_\delta + v_\delta, v_\delta \in B_{\varrho_0}$ with $\|v_\delta\| \leq \delta$ and $C_1||\gamma||_{L^2(\mathbb{R}^3)} > \varepsilon_0$. Thus, it is possible to obtain for all $n \in \mathbb{N}$ and $\delta = \frac{1}{n}$ functions $u_n + v_n, v_n \in B_{\varrho_0}$ such that $||v_n|| \leq \frac{1}{n}$ and $C_1||\gamma_n||_{L^2(\mathbb{R}^3)} > \varepsilon_0$, for $\gamma_n(x) := |f(x,\ \omega_n(x)) - f(x,\ u_n(x))|$, with $\omega_n = u_n + \theta_n v_n$, and $0 \leq \theta_n(x) \leq 1$ depending on u_n and v_n as before. Due to $v_n \to 0$ in E, then $v_n \to 0$ in $L^2(\mathbb{R}^3)$, $v_n \to 0$ a.e. in $\mathbb{R}^3$ and there exists $\psi \in L^2(\mathbb{R}^3)$ such that $|v_n(x)| \leq \psi(x)$ a.e. in $\mathbb{R}^3$. In addition, since $(u_n) \subset B_{\varrho_0}$, it is bounded in E, then $u_n \rightharpoonup u$ in E up to subsequences, then $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$ up to subsequences, hence $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$ and fixed $B_r(0) \subset \mathbb{R}^3$ there exists $\chi_r \in L^2(B_r(0))$ such that $|u_n(x)| \leq \chi_r(x)$ a.e. in $B_r(0)$ up to subsequences. Moreover, $\omega_n \rightharpoonup u$ in E up to subsequences, then $\omega_n(x) \to u$ in $L^2_{loc}(\mathbb{R}^3)$ up to subsequences, thus $\omega_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$, which implies that $\gamma_n(x) \to 0$, a.e. in $\mathbb{R}^3$, provided that f is continuous. Additionally , by Remark 4 with $p = 2$, we get that

$$\begin{aligned} |\gamma_n(x)|^2 &= |f(x,\omega_n(x)) - f(x,u_n(x))|^2 \\ &\leq 2\left[|f(x,\omega_n(x))|^2 + |f(x,u_n(x))|^2\right] \\ &\leq 2\left[C^2|\omega_n(x)|^2 + C^2|u_n(x)|^2\right] \\ &\leq 2\left[C^2|u_n(x)+\theta_n(x)v_n(x)|^2 + C^2|u_n(x)|^2\right] \\ &\leq 2C^2\left[2(|u_n(x)|^2 + |v_n(x)|^2) + |u_n(x)|^2\right] \\ &\leq 2C^2\left[3|u_n(x)|^2 + 2|v_n(x)|^2\right] \\ &\leq 6C^2\left[\chi_r^2(x) + \psi^2(x)\right], \end{aligned} \tag{27}$$

almost everywhere in $B_r(0)$. Since $\chi_r^2 + \psi^2 \in L^1(B_r(0))$, in view of Lebesgue Dominated Convergence Theorem, it shows that

$$\int\limits_{B_r(0)} |\gamma_n(x)|^2 dx \to 0, \text{ as } n \to +\infty. \tag{28}$$

On the other hand, since $(\omega_n) \subset H^1_{rad}(\mathbb{R}^3)$ and $(u_n) \subset H^1_{rad}(\mathbb{R}^3)$ are bounded sequences, applying the characterization of decay of radial functions (see [6, Radial Lemma A.II)] , we know that

$$\lim_{|x|\to+\infty} \omega_n(x) = \lim_{|x|\to+\infty} u_n(x) = 0, \text{ uniformly with respect to } n.$$

Therefore, we give $\zeta > 0$, and there exists $r > 0$ such that $|x| \geq r$ implies $|\omega_n(x)|, |u_n(x)| \leq \zeta$ for all $n \in \mathbb{R}$. Moreover, given $\eta > 0$ by (f_1) there exists $\zeta > 0$ small enough such that $|f(x,s)| \leq \eta|s|$ for all $|s| \leq \zeta$. Hence, for $r > 0$ large enough, one has

$$|f(x,\omega_n(x))| \leq \eta\,|\omega_n(x)| \quad \text{and} \quad |f(x,u_n(x))| \leq \eta\,|u_n(x)|\,,$$

for all $|x| \geq r$ and since (ω_n) and (u_n) are bounded sequences in $L^2(\mathbb{R}^3)$, we obtain that

$$\begin{aligned}\int\limits_{\mathbb{R}^3\setminus B_r(0)} |\gamma_n(x)|^2 dx \leq & 2 \int\limits_{\mathbb{R}^3\setminus B_r(0)} \left[|f(x,\omega_n(x))|^2 + |f(x,u_n(x))|^2\right] dx \\ \leq & 2\eta \int\limits_{\mathbb{R}^3\setminus B_r(0)} \left(|\omega_n(x)|^2 + |u_n(x)|^2\right) dx \\ \leq & 2\eta \sup_n \left(\|\omega_n\|_2^2 + \|u_n\|_2^2\right) \leq C\eta < \frac{1}{2}\left(\frac{\varepsilon_0}{C_1}\right)^2,\end{aligned} \tag{29}$$

for ϑ sufficiently small. Thus, from (28) and (29), we have

$$\left(\frac{\varepsilon_0}{C_1}\right)^2 < \|\gamma_n\|^2_{L^2(\mathbb{R}^3)} = \int\limits_{\mathbb{R}^3} |\gamma_n(x)|^2 dx \leq o_n(1) + \frac{1}{2}\left(\frac{\varepsilon_0}{C_1}\right)^2 \text{ as } n \to +\infty. \tag{30}$$

Hence, letting pass to the limit in (30) as $n \to +\infty$, it yields a contradiction. It shows that B_1 is uniformly differentiable on bounded sets of E.

Next we prove that B_2 is also uniformly differentiable on bounded sets of E. Similarly, note that fixed $\varrho > 0$ and given $u+v, v \in B_\varrho \subset E$, the closed ball centered on the origin, given $\varepsilon > 0$, applying Hardy-Littlewood-Sobolev inequality, Lemma 9 and Lemma 10, there exists absolute constants C_0, C_2, C_3, such that

$$\begin{aligned}&\left|B_2(u+v) - B_2(u) - B_2'(u)v\right| \\ =& \left|\frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_{(u+v)}(u+v)^2 dx - \frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_u u^2 dx - \int\limits_{\mathbb{R}^3} k(x)\phi_u uv dx\right| \\ =& \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \left|\frac{1}{4}k(x)k(y)\frac{u^2(y)u^2(x)}{|x-y|} + \frac{1}{2}k(x)k(y)\frac{u^2(y)v^2(x)}{|x-y|} + k(x)k(y)\frac{u(y)v(y)u(x)v(x)}{|x-y|}\right| dxdy \\ =& \left|\frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_u u^2 dx\right| + \left|\frac{1}{2}\int\limits_{\mathbb{R}^3} k(x)\phi_u v^2 dx\right| + \left|\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} k(x)k(y)\frac{u(y)v(y)u(x)v(x)}{|x-y|} dxdy\right| \\ \leq& C_0\|u\|^4 + C_2\|u\|^2\|v\|^2 + C_3|uv|_{\frac{6}{5}}|uv|_{\frac{6}{5}} \\ \leq& \varepsilon.\end{aligned} \tag{31}$$

Therefore, we finish the proof. ■

4 The linking geometry

In order to prove that I satisfies (I_3) in Theorem 6, as usual, set

$$S = (\partial B_\rho \cap E_1) \quad \text{and} \quad Q = \{re + u_2 : r \geq 0, u_2 \in E_2, \|re + u_2\| \leq r_1\},$$

where $0 < \rho < r_1$ are constants and $e \in E_1, ||e|| = 1$, is chosen suitably. In fact, as a result of the strict inequality in hypothesis (f_3) and by Remark 2, we can select $e \in E_1$ a unitary vector given by the spectral family of operator A and $\varepsilon > 0$ small enough satisfying

$$\begin{aligned}1 =& \|e\|^2 = P_A(e) = \frac{1}{2}(Ae, e)_{L^2(\mathbb{R}^N)} \\ \leq& \frac{1}{2}(\sigma^+ + \varepsilon)\|e\|_2^2 \\ <& \frac{1}{2}a_0\|e\|_2^2 \\ \leq& \frac{1}{2}\int\limits_{\mathbb{R}^3} h(x)e^2(x)dx.\end{aligned} \tag{32}$$

We choose such an e, via (32) it shows that if $r_1 > 0$ is sufficiently large, then $I|_S \geq \alpha > 0$ and $I|_{\partial Q} \leq 0$ for some $\alpha > 0$. Moreover, S and Q "link" (cf. [21]). Therefore, I satisfies (I_3) for some $\alpha > 0$, $\omega = 0$ and arbitrary $\vartheta \in E_2$. Indeed, the following lemma gives the result.

Lemma 12 *Suppose that* $(V_1) - (V_2)$ *on* V *and* $(f_1) - (f_3)$ *on* f *holds, then* I *satisfies* (I_3).

Proof. Note that $S \subset E_1$, then from Remark 4, for $2 < p < 2^*$ and for all $u_1 \in S$. Thus, a direct argument shows that

$$\begin{aligned} I(u_1) &= \frac{1}{2}\|u_1\| - \int\limits_{\mathbb{R}^3} F(x, u_1(x))dx - \frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_{u_1}u_1^2 dx \\ &\geq \frac{1}{2}\rho^2 - \int\limits_{\mathbb{R}^3} (\frac{\varepsilon}{2}|u_1(x)|^2 + \frac{C_\varepsilon}{p}|u_1(x)|^p)dx - \frac{c}{4}\|u_1\|^4 \\ &\geq \frac{1}{2}\rho^2 - (\frac{\varepsilon}{2}C_2^2\|u_1\|^2 + \frac{C_\varepsilon}{p}C_p^p\|u_1\|^p|u_1(x)|^p + \frac{c}{4}\|u_1\|^4) \\ &= \rho^2\left[\frac{1}{2}(1-\varepsilon C_2^2) - (\frac{C_\varepsilon}{p}C_p^p\rho^{p-2} - \frac{c}{4}\rho^2)\right] \\ &\geq \rho^2(b_1 - b_2) = \alpha > 0, \end{aligned} \tag{33}$$

where ε, ρ are chosen small enough, such that $1 > \varepsilon C_2^2$, $\frac{C_\varepsilon}{p}C_p^p\rho^{p-2} > \frac{c}{4}\rho^2$ and also

$$b_1 = \frac{1}{2}(1-\varepsilon C_2^2) > \frac{C_\varepsilon}{p}C_p^p\rho^{p-2} - \frac{c}{4}\rho^2 = b_2.$$

Therefore, from (33), $(I_3)(i)$ holds for I.

With the purpose of proving that I satisfies $(I_3)(ii)$ in Theorem 6, with $\omega = 0$, note that $I(v) \leq 0$, for all $v \in E_2 = E^- \oplus E^0$, then it suffices to show that $I(re+v) \leq 0$ for $r > 0, u \in E_2$ and $||re+v|| \geq r_1$, for some $r_1 > 0$ large enough. Arguing indirectly assume that some sequence $(r_ne + v_n) \subset \mathbb{R}^+e \oplus E_2$ satisfies $||re+v|| \to +\infty$ and $I(re+v) > 0$ for all $n \in \mathbb{N}$. For seeking a contradiction, we can set

$$\tilde{v}_n := \frac{r_ne + v_n}{||r_ne + v_n||} = t_ne + \xi_n,$$

where $t_n \in \mathbb{R}^+, \xi_n = \xi_n^- + \xi_n^0 \in E_2 = E^- \oplus E^0$ and $\|\tilde{v}_n\| = 1$. Provided that $(\tilde{v}_n)$ is bounded, up to subsequences it yields that $\tilde{v}_n \rightharpoonup \tilde{v} = te + \xi$ in E, hence $\tilde{v}_n \to \tilde{v}$ in $L^2_{loc}(\mathbb{R}^3)$. Then, up to subsequences, $\tilde{v}_n(x) \to \tilde{v}(x)$ almost everywhere in $\mathbb{R}^3$, $t_n \to t$ in $\mathbb{R}^+$, $\xi_n^- \rightharpoonup \xi^-$ in E, and $\xi_n^0 \to \xi^0$ in E, since t_n, ξ_n^- and ξ_n^0 are also bounded, $(\xi_n^0) \subset E^0$ and E^0 is finite dimensional. Noting that

$$1 = ||t_ne + \xi_n||^2 = t_n^2 + ||\xi_n^-||^2 + ||\xi_n^0||^2,$$

it follows that $0 \leq t_n^2 \leq 1$, and it yields

$$\begin{aligned} \frac{I(r_ne+v_n)}{||r_ne+v_n||^2} &= t_n^2||e||^2 - ||\xi_n^-||^2 - \int\limits_{\mathbb{R}^3} \frac{F(x, r_ne(x) + v_n(x))}{||r_ne+v_n||^2}dx - \frac{1}{4}\int\limits_{\mathbb{R}^3} \frac{k(x)\phi_u u^2}{||r_ne+v_n||^2}dx \\ &= t_n^2 - 1 + \frac{1}{2}||\xi_n^0||^2 - \int\limits_{\mathbb{R}^3} \frac{F(x, r_ne(x) + v_n(x))}{||r_ne+v_n||^2}dx - \frac{1}{4}\int\limits_{\mathbb{R}^3} \frac{k(x)\phi_u u^2}{||r_ne+v_n||^2}dx > 0, \end{aligned} \tag{34}$$

hence $0 < t \leq 1$. Moreover, from (32) it is possible to choose a bounded domain $\Omega \subset \mathbb{R}^3$, such that

$$1 < \int\limits_{\Omega} h(x)e^2(x)dx.$$

Then,

$$\begin{aligned} 0 &> \frac{t^2}{2} - \frac{t^2}{2}\int_{\Omega} h(x)e^2(x)dx \\ &\geq t^2\left(\frac{1}{2} - \frac{1}{2}\int_{\Omega} h(x)e^2(x)dx\right) - \left(1 - \frac{1}{2}||\xi^0||^2 - \frac{t^2}{2}\right) - \frac{1}{2}\int_{\Omega} h(x)\xi^2(x)dx \\ &= t^2\left(1 - \frac{1}{2}\int_{\Omega} h(x)e^2(x)dx\right) - 1 + \frac{1}{2}||\xi^0||^2 - \frac{1}{2}\int_{\Omega} h(x)\xi^2(x)dx. \end{aligned} \tag{35}$$

On the other hand, from assumptions $(f_1)-(f_2)$ and since $\tilde{v}_n$ is convergent in $L^2(\Omega)$, there exists some $\psi \in L^1(\Omega)$ such that

$$\left|\frac{F(\cdot,\ r_n e(\cdot) + v_n(\cdot))}{||r_n e + v_n||^2}\right| \leq r_\infty |\tilde{v}_n(\cdot)|^2 \leq \psi(\cdot) \in L^1(\Omega).$$

Moreover, provided that $||r_n e + v_n|| \to +\infty$ and $\tilde{v}_n(x) \to \tilde{v}(x) \neq 0$, almost everywhere in $supp(\tilde{v})$, we infer that $v_n(x) = \tilde{v}_n(x)\,||r_n e(x) + v_n(x)|| \to +\infty$ almost everywhere in $supp(\tilde{v})$, as $n \to +\infty$. Furthermore, we get that

$$\frac{F(x,\ r_n e(x) + v_n(x))}{||r_n e + v_n||^2} = \frac{F(x,\ \tilde{v}_n(x)\,||r_n e + u_n||)\tilde{v}_n^2(x)}{\tilde{v}_n^2(x)\,||r_n e + v_n||^2} \to \frac{1}{2}h(x)\tilde{v}^2(x),$$

almost everywhere in $supp(\tilde{v})$ as $n \to +\infty$. Note that, $supp(\tilde{v}) \neq \varnothing$, since $\tilde{v} = te + \xi$, with $supp(e) \neq \varnothing$ and $(e,\xi)_{L^2(\mathbb{R}^3)} = 0$. Thus, because of Lebesgue Dominated Convergence Theorem, we conclude that

$$\int_{\Omega} \frac{F(x,\ r_n e(x) + v_n(x))}{||r_n e + v_n||^2}dx \to \frac{1}{2}\int_{\Omega} h(x)(te(x) + \xi(x))^2 dx \quad \text{as } n \to +\infty.$$

From (34) one has

$$t_n^2 - 1 + \frac{1}{2}||\xi_n^0||^2 - \int_{\mathbb{R}^3} \frac{F(x, r_n e(x) + v_n(x))}{||r_n e + v_n||^2}dx > 0.$$

Passing to the limit as $n \to +\infty$, one infers that

$$\begin{aligned} 0 &\leq t^2 - 1 + \frac{1}{2}||\xi^0||^2 - \frac{1}{2}\int_{\Omega} h(x)(t^2 e^2(x) + \xi^2(x))dx \\ &= t^2\left(1 - \frac{1}{2}\int_{\Omega} h(x)e^2(x)dx\right) - 1 + \frac{1}{2}||\xi^0||^2 - \frac{1}{2}\int_{\Omega} h(x)\xi^2(x)dx. \end{aligned} \tag{36}$$

This contradicts (35). Therefore we complete the proof. ■

5 The boundedness of Cerami sequences

The following lemma ensures I satisfies last hypothesis in Theorem 6. Finally, with this result, it is possible to prove Theorem 1. For the boundedness of Cerami sequences, standard arguments are applied and hypotheses (f_4) and (f_5) are used. It is important to point out that these assumptions are merely used in order to prove next lemma, since the special properties of radial functions are not sufficient when problem (1) is treated in $\mathbb{R}^3$.

Lemma 13 *Assuming that* V *satisfies* $(V_1)-(V_2)$ *and* f *satisfies* $(f_1)-(f_5)$*, then* I *satisfies* (I_4).

Proof. We may choose $b > 0$ is an arbitrary constant, and take $(u_n) \subset I^{-1}([c-b, c+b])$ satisfies $(1+||u_n||)||I'(u_n)|| \to 0$, it is necessary to show that (u_n) is bounded. Suppose by contradiction that $||u_n|| \to +\infty$, up to subsequences. We select $\bar{u}_n := \frac{u_n}{||u_n||}$, it is bounded. According to the compact embeddings previously mentioned (cf. [29] and [6]). Thus

$\bar{u}_n \rightharpoonup \bar{u}$ in E and $\bar{u}_n \to \bar{u}$ in $L^\beta(\mathbb{R}^3)$, for $\beta \in (2, 2^*)$. Then we define $u_n = u_n^+ + u_n^- + u_n^0 \in E^+ \oplus E^- \oplus E^0$, and we easily get that

$$\begin{aligned} o_n(1) &= I'(u_n)\frac{u_n^+}{||u_n||^2} = \frac{1}{||u_n||} I'(u_n)\bar{u}_n^+ \\ &= ||\bar{u}_n^+||^2 - \int_{\mathbb{R}^3} \frac{f(x,\ u_n(x))}{u_n(x)} \bar{u}_n(x)\,\bar{u}_n^+(x)dx - \int_{\mathbb{R}^3} \frac{k(x)\phi_{u_n} u_n(x)\bar{u}_n^+(x)}{\|u_n\|}dx \\ &= ||\bar{u}_n^+||^2 - \int_{\mathbb{R}^3} \frac{f(x,\ u_n(x))}{u_n(x)} (\bar{u}_n^+(x))^2 dx - \int_{\mathbb{R}^3} k(x)\phi_{u_n} (\bar{u}_n^+(x))^2 dx, \end{aligned} \tag{37}$$

and

$$\begin{aligned} o_n(1) &= I'(u_n)\frac{u_n^-}{||u_n||^2} = \frac{1}{||u_n||} I'(u_n)\bar{u}_n^- \\ &= -||\bar{u}_n^-||^2 - \int_{\mathbb{R}^3} \frac{f(x,\ u_n(x))}{u_n(x)} \bar{u}_n(x)\,\bar{u}_n^-(x)dx - \int_{\mathbb{R}^3} \frac{k(x)\phi_{u_n} u_n(x)\bar{u}_n^-(x)}{\|u_n\|}dx \\ &= -||\bar{u}_n^-||^2 - \int_{\mathbb{R}^3} \frac{f(x,\ u_n(x))}{u_n(x)} (\tilde{u}_n^-(x))^2 dx - \int_{\mathbb{R}^3} k(x)\phi_{u_n} (\bar{u}_n^-(x))^2 dx. \end{aligned} \tag{38}$$

Subtracting (38) from (37), and by means of $1 = ||\bar{u}_n^+||^2 + ||\bar{u}_n^-||^2 + ||\bar{u}_n^0||^2$. A direct computation shows that

$$o_n(1) = 1 - ||\bar{u}_n^0||^2 - \int_{\mathbb{R}^3} \frac{f(x,\ u_n(x))}{u_n(x)} [(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2]dx - \int_{\mathbb{R}^3} k(x)\phi_{u_n} [(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2]dx. \tag{39}$$

Provided that $(\bar{u}_n^0) \subset E^0$, which is finite dimensional, then the weak convergence implies that $\bar{u}_n^0 \to \bar{u}^0$ in E. On the other hand, given $\varphi \in C_0^\infty(\mathbb{R}^3)$ and setting $supp\,(\varphi) = K$, since $\bar{u}_n \to \bar{u}$ in $L^2(K)$, and applying Lebesgue Dominated Convergence Theorem, one obtains that

$$\int_K \frac{f(x,\ u_n(x))}{u_n(x)} \bar{u}_n(x)\,\varphi(x)dx = \int_K h(x)\bar{u}_n(x)\varphi(x)\,dx + o_n(1), \text{ as } n \to +\infty.$$

Hence, it yields that

$$\begin{aligned} o_n(1) &= \frac{I'(u_n)\varphi}{||u_n||} \\ &= \frac{P_A'(u_n)\varphi}{||u_n||} - \int_K \frac{f(x, u_n(x))}{u_n(x)} \bar{u}_n(x)\varphi(x)dx - \int_K k(x)\phi_{u_n}\bar{u}_n(x)\varphi(x)\,dx \\ &= (A\bar{u}_n, \varphi)_{L^2(\mathbb{R}^3)} - \int_K (h(x) + k(x)\phi_{u_n})\bar{u}_n(x)\,\varphi(x)\,dx + o_n(1) \\ &= (\mathcal{J}\bar{u}_n, \varphi)_{L^2(\mathbb{R}^3)} + o_n(1) = (\mathcal{J}\bar{u}, \varphi)_{L^2(\mathbb{R}^3)} + o_n(1). \end{aligned} \tag{40}$$

In view of (40), if $\bar{u} \neq 0$, it is an eigenvector of $\mathcal{J}$ with eigenvalue 0. Nevertheless, by (f_4), $0 \notin \sigma_p(\mathcal{J})$ and hence we have $\tilde{u} = 0$. It deduces that $\bar{u}^+ = \bar{u}^- = \bar{u}^0 = 0$ and thus $\bar{u}_n^\pm \to 0$ in $L^\beta(\mathbb{R}^3)$, for $\beta \in (2, 2^*)$, in view of the compact embeddings, and $\bar{u}_n^0 \to 0$ in E. Therefore, we infer from (39) that

$$\int_{\mathbb{R}^3} \frac{f(x,\ u_n(x))}{u_n(x)} [(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2]dx + \int_{\mathbb{R}^3} k(x)\phi_{u_n} [(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2]dx = 1 + o_n(1). \tag{41}$$

Now, we definine $\Omega_n := \left\{x \in \mathbb{R}^3 : \frac{|f(x,u_n(x))|}{|u_n(x)|} < \kappa_0\right\} \subset \mathbb{R}^3$ for all $n \in \mathbb{N}$, $b_0 > 0$ and for $\kappa_0 > 0$ given by (f_5) and then one infers that

$$\begin{aligned}&\int\limits_{\mathbb{R}^3\backslash\Omega_n} \frac{|f(x,u_n(x))|}{|u_n(x)|}\left|(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2\right|dx + \int\limits_{\mathbb{R}^3\backslash\Omega_n}\left|k(x)\phi_{u_n}[(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2]\right|dx \\ &\geq \int\limits_{\mathbb{R}^3\backslash\Omega_n} \frac{|f(x,u_n(x))|}{|u_n(x)|}\left|(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2\right|dx \geq \kappa_0 b_0 + o_n(1).\end{aligned} \tag{42}$$

Thus we have that

$$\liminf_{n\to+\infty} \int\limits_{\mathbb{R}^3\backslash\Omega_n} \frac{|f(x,u_n(x))|}{|u_n(x)|}\left|(\bar{u}_n^+(x))^2 - (\bar{u}_n^-(x))^2\right|dx \geq \kappa_0 b_0. \tag{43}$$

Furthermore, due to $\left|\frac{f(x,s)}{s}\right|$ is bounded, applying Hölder Inequality for $\beta \in (2, 2^*)$ and for some $C > 0$, then one has that

$$\int\limits_{\mathbb{R}^3\backslash\Omega_n} \frac{|f(x,u_n(x))|}{|u_n(x)|}\left|(\bar{u}_n^+(x))^2 - (\tilde{u}_n^-(x))^2\right|dx \leq C\left|\mathbb{R}^3\backslash\Omega_n\right|^{\frac{\beta-2}{\beta}} \left\|\bar{u}_n^+ + \bar{u}_n^-\right\|^2_{L^\beta(\mathbb{R}^3)}. \tag{44}$$

Provided that $\bar{u}_n^\pm \to 0$ in $L^\beta(\mathbb{R}^3)$ for $\beta \in (2, 2^*)$, and hence it follows from (43) and (44) that

$$\left|\mathbb{R}^3\backslash\Omega_n\right| \to +\infty, \quad \text{as} \quad n \to +\infty. \tag{45}$$

On the other hand, in view of (f_5) and since (u_n) is a Cerami sequence, we can find a constant $M_0 > 0$ such that

$$\begin{aligned} M_0 &\geq I(u_n) - \frac{1}{2}I'(u_n)u_n \\ &= \int\limits_{\mathbb{R}^3} T(x,u_n(x))dx + \frac{1}{4}\int\limits_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2 dx \\ &\geq \int\limits_{\mathbb{R}^3\backslash\Omega_n} T(x,u_n(x))dx \\ &\geq \kappa_0\left|\mathbb{R}^3\backslash\Omega_n\right|, \end{aligned} \tag{46}$$

which contradicts (45). Therefore, (u_n) is bounded and the result holds. ■

Finally, we prove the main result of this section.

Proof of Theorem 1.. Provided that I satisfies all assumptions (I_1)-(I_4) in Theorem 6, it ensures a critical point $u \in E$ of I , with $I(u) = c \geq \alpha > 0$, hence u is a non-trivial critical point of $I : E \to \mathbb{R}$. It represents that $I(u)v = 0$, for all $v \in H^1_{rad}(\mathbb{R}^3)$. Nevertheless, the Principle of Symmetric Criticality [24] implies that $I(u)v = 0$ for all $v \in H^1(\mathbb{R}^3)$, in other words, u is a critical point of I as a functional defined on the whole $H^1(\mathbb{R}^3)$. Since $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$, it yields that u is a nontrivial weak solution for (1). Additionally, because of $u \in E$, it is a radial weak solution. ■

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References

[1] A. Ambrosetti, D. Ruiz. Multiple bound states for the Schröldinger-Poisson problem. *Commun. Contemp. Math.*, 10(2008): 391–404.

[2] A. Ambrosetti. On Schrödinger-Poisson systems. *Milan J. Math.*, 76(2008): 257–274.

[3] A. Ambrosetti, A. Malchiodi. Perturbation methods and semilinear elliptic problem on $\mathbb{R}^N$, In: Progress in Mathematics, vol. 240. Birkhäuser, Boston, 2005.

[4] A. Azzollini, A. Pomponio. Ground state solutions for the nonlinear Schrödinger-Maxwell equations. *J. Math. Anal. Appl.*, 345(2008): 90–108.

[5] V. Benci, D. Fortunato. An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, 11(1998): 283–293.

[6] H. Berestycki, P.-L. Lions. Nonlinear scalar field equations I. *Arch. Ration. Mech. Anal.*, 82(1983): 313–346.

[7] G. Cerami, R. Molle. Positive bound state solution for some Schrödinger-Poisson systems. *Nonlinearity*, 29(2016): 3103–3119.

[8] G. Cerami, G. Vaira. Positive solutions for some non-autonomous Schrödinger-Poisson systems. *J. Differential Equations*, 248(2010): 521–543.

[9] G.-M. Coclite. A multiplicity result for the nonlinear Schrödinger-Maxwell equations. *Commun. Appl. Anal.*, 7(2003): 417–423.

[10] T. D'Aprile, D. Mugnai. Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations. *Proc. Roy. Soc. Edinb. Sect. A*, 134(2004): 893–906.

[11] T. D'Aprile, D. Mugnai. Non-existence results for the coupled Klein-Gordon-Maxwell equations. *Adv. Nonlinear Stud.*, 4(2004): 307–322.

[12] T. D'Aprile, J. Wei. On bound states concentrating on spheres for the Maxwell-Schrödinger equation. *SIAM J. Math. Anal.*, 37(2005): 321–342.

[13] P. D'Avenia, A. Pomponio, G. Vaira. Infinitely many positive solutions for a Schrödinger-Poisson system. *Nonlinear Anal.*, 74(2011): 5705–5721.

[14] M. Ghimenti, J. Van Schaftingen. Nodal solutions for the Choquard equation. *J. Funct. Anal.*, 271(2016): 107–135.

[15] X.-M. He. Multiplicity and concentration of positive solutions for the Schrödinger-Poisson equations. *Z. Angew. Math. Phys.*, 5(2011): 869–889.

[16] P.-D. Hislop, I.-M. Sigal. Introduction to Spectral Theory with Applications to Schrödinger Equations, SpringerVerlag, New York, Inc., 1996.

[17] I. Ianni. Solutions of the Schrödinger-Poisson problem concentrating on spheres, Part II: Existence. *Math. Models Methods Appl. Sci.*, 19(2009): 877–910.

[18] I. Ianni, G. Vaira, Solutions of the Schrödinger-Poisson problem concentrating on spheres, Part I: Necessary condition. *Math. Models Methods Appl. Sci.*, 19(2009): 707–720.

[19] G.-B. Li, S.-J. Peng, C.-H. Wang, Multi-bump solutions for the nonlinear Schrödinger-Poisson system. *J. Math. Phys.*, 52(2011): 053505.

[20] E.-H. Lieb and M. Loss. Analysis,volume 14 of Graduate Studies in mathematics, American Mathematical Society, Providence, RI, second edition, 2001.

[21] L.-A. Maia, M. Soares. An abstract linking theorem applied to indefinite problems via spectral properties. *Advanced Nonlinear Studies*, 19(2019): 1–23.

[22] L.-A. Maia, M. Soares. Spectral Theory Approach for a Class of Radial Indefinite Variational Problems. *J. Differential Equations*, 266(2019): 6905–6923.

[23] C.-R. Oliveira. Intermediate Spectral Theory and Quantum Dynamics, Progress in Mathematical Physics, vol. 54, Birkhuser, 2009.

[24] R.-S. Palais. The principle of symmetric criticality. *Comm. Math. Phys.*, 69(1979): 19–30.

[25] D. Ruiz. The Schrödinger-Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.*, 237(2006): 655–674.

[26] D. Ruiz, Semiclassical states for coupled Schrödinger-Maxwell equations concentration around a sphere. *Math. Models Methods Appl. Sci.*, 15(2005): 141–164.

[27] D. Ruiz, G. Vaira,Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of potential. *Rev. Mat. Iberoam.*, 27(2011): 253–271.

[28] O. Sánchez, J. Soler. Long-time dynamics of the Schrödinger-Poisson-Slater system. *J. Stat. Phys.*, 114(2004): 179–204.

[29] W.-A. Strauss. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.*, 55(1977): 149–162.

[30] C.-A. Stuart, H.-S. Zhou. Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$. *Comm. Partial Differential Equations*, 24(1999): 1731–1758.

[31] J. Wang, L.-X. Tian, J.-X. Xu, F.-B. Zhang. Existence and concentration of positive ground state solutions for

semilinear Schrödinger-Poisson systems. *Adv. Nonlinear Stud.*, 13(2013): 553–582.

[32] J. Wang, L.-X. Tian, J.-X. Xu, F.-B. Zhang. Existence and concentration of positive solutions for semilinear Schrödinger-Poisson systems in $\mathbb{R}^3$. *Calc. Var. Partial Differ. Equ.*, 48(2013): 243–273.

[33] J. Wang, L.-X. Tian, J.-X. Xu, F.-B. Zhang. Existence of multiple positive solutions for Schrödinger-Poisson systems with critical growth. *Z. Angew. Math. Phys.*, 66(2015): 2441–2471.

[34] Y.-W. Ye, C.-L. Tang. Existence and multiplicity of solutions for Schrödinger-Poisson equations with sign-changing potential. *Calc. Var. PDE*, 53(2015): 383–411.

[35] L. Zhao, H. Liu, F. Zhao. Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential. *J. Differ. Equ.*, 255(2013): 1–23.

[36] L.-G. Zhao, F.-K. Zhao. On the existence of solutions for the Schrödinger-Poissom equations. *J. Math. Anal. Appl.*, 346(2008): 155–169.