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Stability Analysis for a Reaction-Diffusion Equation with Spatio-temporal Delay

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Abstract: In this paper, we propose a reaction-diffusion equation with spatio-temporal delay and Dirichlet boundary condition to model the single population dynamics. By using the implicit function theorem we establish the existence of a positive spatially non-homogeneous equilibrium for the model. Meanwhile, we prove that the positive spatially non-homogeneous equilibrium can bifurcate from the trivial equilibrium. Under certain conditions it is founded that for given spatiotemporal delay, the bifurcated positive spatially non-homogeneous equilibrium is stable and the Hopf bifurcation cannot occur.

Keywords: Reaction-diffusion equation; Spatio-temporal delay; Stability analysis; Dirichlet boundary condition

1 Introduction

Reaction-diffusion equations with spatio-temporal delay has been one of hot topics of recent biological mathematics and many interesting results on the stability and bifurcation have been established [1–4]. Since individuals move randomly and may be at different points at the different time, the diffusion and the delay are not independent of each other, incorporating the so-called spatio-temporal delay into the reaction-diffusion equation is more close to the reality. On such assumption, Britton [5] initially introduced the spatio-temporal delay term into the diffusive Fisher equation on the infinite domain

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u(x,t)}{\partial x^2} + [u(x,t)(1+\alpha-\beta u^2 - (1+\alpha-\beta)g**u)], \quad x\in\Omega, \quad t>0,$$

where

$$g**u = \int_{-\infty}^{t}\int_{\Omega} g(x,y,t-s)u(y,s)dyds.$$

Here the function $g(x,y,t)$ is a general spatio-temporal average, which has been studied extensively in recent years. Gourley and Britton [6, 7] studied a predator-prey system with spatio-temporal delay. Li, Ruan [8] and Wang and Li [9] have established the existence and stability of traveling wave of nonlocal reaction-diffusion equation. Recently, Chen and Yu [10] considered a diffusive logistic model incorporating a class of spatio-temporal delay on a bounded domain with the Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda u(x,t)F\Big(u(x,t), \displaystyle\int_0^{\infty}\int_0^{\pi} G(x,y,s)f(s)u(y,t-s)dyds\Big), \quad x\in(0,\pi), \quad t>0, \\ u(0,t)=u(\pi,t)=0, \quad t>0, \end{cases}$$

where λ, d are positive constants and $G(x,y,t)$ is the solution of the following equation

$$\begin{cases} \dfrac{\partial G}{\partial t} = d\dfrac{\partial^2 G}{\partial y^2}, \\ G=0, \quad y=0,\pi, \\ G(x,y,0)=\delta(x-y). \end{cases}$$

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They proved that the nonhomogeneous steady state is locally asymptotically stable and Hopf bifurcations can not occur. Zuo and Song [11] studied spatio-temporal patterns of a general reaction-diffusion equation with nonlocal delay under the Neumann boundary condition. For the reaction-diffusion equation with spatio-temporal delay under the Dirichlet boundary conditions, we can refer to [12–15].

Based on the work above, we analyze the following reaction-diffusion equation with spatio-temporal delay and homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda u(x,t)F\Big(u(x,t), \displaystyle\int_0^\infty \int_0^l \kappa(y,s) f(u(y,t-s))dyds\Big), x \in (0,l), t > 0, \\ u(0,t) = u(l,t) = 0, \qquad t > 0, \end{cases} \tag{1.1}$$

where λ is a positive constant, d is a positive diffusion coefficient, κ is a nonnegative continuous kernel function and $\int_0^\infty \int_0^l \kappa(y,s)dyds = 1$, f is Lipschitz continuous on any compact interval in R_+ with $f(x) > 0, x > 0$ and $f(0) = 0$, $f'(0) > 0$. Throughout the paper, we suppose that the function $F(x,y)$ is smooth, $F(0,0) = 1$, denote

$$a = \frac{\partial F}{\partial x}(0,0), b = \frac{\partial F}{\partial y}(0,0). \tag{1.2}$$

In addition, $X = H^2 \bigcap H_0^1$, $Y = L^2$. We also define the complexification of Z to be $Z_C := Z \bigoplus iZ = \{x_1 + ix_2 \mid x_1, x_2 \in Z\}$. For the complex-valued Hilbert space Y_C, we use the standard inner product $\langle u, v\rangle = \int_0^l u(x)\overline{v}(x)dx$.

This paper is organized as follows. In Section 2, we apply the implicit function theorem to prove the existence of the positive equilibrium. In Section 3, we investigate the stability of bifurcated positive equilibrium. In Section 4, we make a brief conclusion.

2 Existence of the positive equilibrium solution

We first study the existence of the positive steady state solution of equation (1.1), which is a solution of the following elliptic equation

$$\begin{cases} d\dfrac{\partial^2}{\partial x^2} + \lambda u(x)F\Big(u(x), \displaystyle\int_0^\infty \int_0^l \kappa(y,s) f(u(y))dyds\Big) = 0, \quad x \in (0,l), \\ u(0) = u(l) = 0. \end{cases} \tag{2.1}$$

It is well-known that the following decompositions satisfy

$$Y = L^2[0,l] = \mathcal{N}\Big(d\frac{\partial^2}{\partial x^2} + \lambda_0\Big) \bigoplus \mathcal{R}\Big(d\frac{\partial^2}{\partial x^2} + \lambda_0\Big), \tag{2.2}$$

where

$$\lambda_0 = d(\frac{\pi}{l})^2, \quad \mathcal{N}(d\frac{\partial^2}{\partial x^2} + \lambda_0) = span\{\sin\frac{\pi}{l}(\cdot)\},$$

$$\mathcal{R}\Big(d\frac{\partial^2}{\partial x^2} + \lambda_0\Big) = \Big\{y \in L^2[0,l] : \langle \sin\frac{\pi}{l}(\cdot), y\rangle \triangleq \int_0^l \sin\frac{\pi}{l}(x)\overline{y}(x)dx = 0\Big\}.$$

Similar to [1, 4], we use the implicit function theorem to prove the existence of the positive equilibrium near $\lambda = \lambda_0$.

Theorem 1 *Suppose that a, b and $f'(0)$ satisfy*

$$a + bf'(0) < 0, \tag{2.3}$$

where a and b are defined as in (1.2). Then there exists $\lambda^* > \lambda_0$, and a continuously differentiable mapping $\lambda \mapsto (\xi_\lambda, \alpha_\lambda)$ from $[\lambda_0, \lambda^*]$ to $X \bigcap \mathcal{R}\Big(d\frac{\partial^2}{\partial x^2} + \lambda_0\Big) \times \mathcal{R}^+$ such that equation (1.1) has a positive equilibrium solution

$$u_\lambda = \alpha_\lambda(\lambda - \lambda_0)\big[\sin\big(\frac{\pi x}{l}\big) + (\lambda - \lambda_0)\xi_\lambda\big] \tag{2.4}$$

for $\lambda \in (\lambda_0, \lambda^*]$. Moreover

$$\alpha_{\lambda_0} = -\frac{\int_0^l sin^2\big(\frac{\pi x}{l}\big)dx}{\lambda_0[a + bf'(0)]\int_0^l sin^3\big(\frac{\pi x}{l}\big)dx}, \tag{2.5}$$

and $\xi_{\lambda_0} \in X$ *is the unique solution of the equation*

$$(d\frac{\partial^2}{\partial x^2}+\lambda_0)\xi+sin(\frac{\pi x}{l})+\lambda_0\alpha_{\lambda_0}(a+bf'(0))sin^2(\frac{\pi x}{l})=0. \tag{2.6}$$

Proof. Since a, b and $f'(0)$ satisfy equation (2.3), we obtain that α_{λ_0} is well defined and positive. Due to the operator $d\frac{\partial^2}{\partial x^2}+\lambda_0$ being bijective from $X\bigcap\mathcal{R}\big(d\frac{\partial^2}{\partial x^2}+\lambda_0\big)$ to $\mathcal{R}\big(d\frac{\partial^2}{\partial x^2}+\lambda_0\big)$, we can define a mapping $m: X\times R^2\mapsto Y\times R$ by

$$m(\xi,\alpha,\lambda)=\big(d\frac{\partial^2}{\partial x^2}+\lambda_0\big)\xi+sin(\frac{\pi x}{l})+(\lambda-\lambda_0)\xi+\lambda[sin(\frac{\pi x}{l})+(\lambda-\lambda_0\xi)]m_1(\xi,\alpha,\lambda), \tag{2.7}$$

where

$$m_1(\xi,\alpha,\lambda)=\begin{cases}\dfrac{F(u_\lambda,\int_0^\infty\int_0^l\kappa(y,s)f(u_\lambda)dyds)-1}{\lambda-\lambda_0}, & \lambda\neq\lambda_0,\\ \alpha a\sin\big(\dfrac{\pi x}{l}\big)+\alpha bf'(0)\sin\big(\dfrac{\pi x}{l}\big), & \lambda=\lambda_0,\end{cases} \tag{2.8}$$

for $u_\lambda=\alpha(\lambda-\lambda_0)\big[\sin\big(\frac{\pi x}{l}\big)+(\lambda-\lambda_0)\xi\big]$.

Noticing that $\int_0^\infty\int_0^l\kappa(y,s)dyds=1$ and $f(0)=0$, when $\lambda=\lambda_0$, we have

$$m(\xi,\alpha,\lambda_0)=\big(d\frac{\partial^2}{\partial x^2}+\lambda_0\big)\xi+\sin\big(\frac{\pi x}{l}\big)+\alpha\lambda_0(a+bf'(0))\sin^2\big(\frac{\pi x}{l}\big). \tag{2.9}$$

From the definition of ξ_{λ_0}, we have that $m(\xi_{\lambda_0},\alpha_{\lambda_0},\lambda_0)=0$. And the Fréchet derivative of m with respect to (ξ,α) at $(\xi_{\lambda_0},\alpha_{\lambda_0},\lambda_0)$ is

$$D_{(\xi,\alpha)}m(\xi_{\lambda_0},\alpha_{\lambda_0},\lambda_0)(\eta,\epsilon)=\big(d\frac{\partial^2}{\partial x^2}+\lambda_0\big)\eta+\epsilon\lambda_0(a+bf'(0))\sin^2(\frac{\pi x}{l}). \tag{2.10}$$

Since a, b and $f'(0)$ satisfy equation (2.3), we see that $D_{(\xi,\alpha)}m(\xi_{\lambda_0},\alpha_{\lambda_0},\lambda_0)$ is bijective from $X\times R$ to $Y\times R$. And the implicit function theorem implies that there exists a $\lambda^*>\lambda_0$, and a continuously differentiable mapping $\lambda\mapsto(\xi_\lambda,\alpha_\lambda)\in X\times R^+$ such that

$$m(\xi_\lambda,\alpha_\lambda,\lambda)=0, for\quad \lambda\in[\lambda_0,\lambda^*].$$

Substituting $u_\lambda=\alpha_\lambda(\lambda-\lambda_0)[\sin(\frac{\pi x}{l})+(\lambda-\lambda_0)\xi_\lambda]\in X$ into equation (2.1), we can find that it satisfies equation (2.1). The proof is completed. ■

3 Stability analysis of the positive equilibrium solution

In the following, we will analyze the linear stability of u_λ and always assume that a, b and $f'(0)$ satisfy equation (2.3). Moreover, we assume that $\lambda\in(\lambda_0,\lambda^*]$ unless otherwise specified.

The linearization of equation (1.1) at u_λ is given by

$$\begin{cases}\dfrac{\partial v(x,t)}{\partial t}=A(\lambda)v(x,t)+\lambda u_\lambda h_1(x)v(x,t)+\lambda u_\lambda h_2(x)\displaystyle\int_0^\infty\int_0^l\kappa(y,s)f(v(y,t-s))dyds, & t>0,\\ v(0,t)=v(l,t)=0,\end{cases} \tag{3.1}$$

where $A(\lambda): D(A(\lambda))\mapsto Y$ is a linear operator defined by

$$A(\lambda)=d\frac{\partial^2}{\partial x^2}+\lambda F(u_\lambda,\int_0^\infty\int_0^l\kappa(y,s)f(u_\lambda(y))dyds) \tag{3.2}$$

with domain $D(A(\lambda))=X$ and

$$\begin{cases}h_1(x)=\dfrac{\partial F}{\partial x}\big(u_\lambda,\displaystyle\int_0^\infty\int_0^l\kappa(y,s)f(u_\lambda(y))dyds\big),\\ h_2(x)=\dfrac{\partial F}{\partial y}\big(u_\lambda,\displaystyle\int_0^\infty\int_0^l\kappa(y,s)f(u_\lambda(y))dyds\big).\end{cases} \tag{3.3}$$

We define the set of eigenvalues $S(\lambda)$ of system (3.1) by

$$S(\lambda) = \{\mu \in C : \Delta(\lambda, \mu)\psi = 0, \text{ for some } \psi \in X_C \setminus \{0\}\}, \tag{3.4}$$

where

$$\Delta(\lambda, \mu)\psi := A(\lambda)\psi + \lambda u_\lambda h_1(x)\psi + \lambda u_\lambda h_2(x)\int_0^\infty \int_0^l \kappa(y,s)f(\psi(y)e^{-\mu s})dyds - \mu\psi.$$

In the following, we will determine the stability of the steady state solution u_λ by the profile of the set $S(\lambda)$.

Theorem 2 *Suppose that the constants a, b and $f'(0)$ satisfy equation (2.3) and $f(x) \leq x$ for $x \geq 0$, then there exists $\lambda_1 > \lambda_0$, where $\lambda_1 \leq \lambda^*$, such that for any $\lambda \in (\lambda_0, \lambda_1]$,*

$$S(\lambda) \subset \{x + iy : x, y \in R, x < 0\}. \tag{3.5}$$

Proof. By contradiction, we suppose that there exists a sequence $\{\lambda^n\}$, $n \in \mathbb{N}^+$, such that $\lambda^n > \lambda_0$ and $\lim_{n\to\infty} \lambda^n = \lambda_0$. For any $n \in \mathbb{N}^+$, the corresponding eigenvalue problem

$$\begin{cases} A(\lambda^n)\psi + \lambda^n u_{\lambda^n} h_1(x)\psi + \lambda^n u_{\lambda^n} h_2(x)\displaystyle\int_0^\infty \int_0^l \kappa(y,s)f(\psi(y)e^{-\mu s})dyds = \mu\psi, x \in (0,l), \\ \psi(0) = \psi(l) = 0, \end{cases} \tag{3.6}$$

has an eigenvalue μ_{λ^n} with a nonnegative real part. In the following, we will obtain a contradiction about this suppose.

Without loss of generality, we assume the associated eigenfunction ψ_{λ^n} with respect to μ_{λ^n} satisfying $\| \psi_{\lambda^n} \|_{Y_C} = 1$. For each $n \in \mathbb{N}^+$, ψ_{λ^n} can be decomposed as $\psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$, where $c_{\lambda^n} \in \mathcal{C}$, u_{λ_n} is the positive solution of equation (1.1) for $\lambda = \lambda_n$ and satisfies equation (2.4), and $\phi_{\lambda_n} \in X_C$ satisfies $\langle \phi_{\lambda_n}, u_{\lambda_n} \rangle = 0$.

Since

$$A(\lambda^n)u_{\lambda^n} = 0$$

and

$$\langle A(\lambda^n)\phi_{\lambda_n}, u_{\lambda^n} \rangle = \langle \phi_{\lambda_n}, A(\lambda^n)u_{\lambda^n} \rangle,$$

substituting $\mu = \mu_{\lambda^n}$ and $\psi = \psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$ into the first equation of (2.17), multiplying the inner product with $\psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$, we obtain that

$$\langle A(\lambda^n)\phi_{\lambda_n}, \phi_{\lambda_n} \rangle = -T_{\lambda^n} + \mu_{\lambda^n}, \tag{3.7}$$

where

$$T_{\lambda^n} = \lambda^n \langle \psi_{\lambda^n}, \lambda^n u_{\lambda^n} h_1(x)\psi_{\lambda_n} \rangle + \lambda^n \langle \psi_{\lambda^n}, u_{\lambda^n} h_2(x)\int_0^\infty \int_0^l \kappa(y,s)f(\psi_{\lambda_n}(y)e^{-\mu_{\lambda_n} s})dyds \rangle.$$

For $\lambda \in (d, \lambda_1]$, $u_\lambda(x)$ is bounded which implies that $\| h_i(x) \|_\infty < C$ $(i = 1, 2)$ for the positive number $C > 0$. Noting that μ_{λ^n} has a nonnegative real part, we have that

$$\begin{aligned} |T_{\lambda^n}| &\leq C \| u_{\lambda^n} \|_\infty \max\{\lambda^n\} \| \psi_{\lambda^n} \|^2 + C \| u_{\lambda^n} \|_\infty \max\{\lambda^n\} \| \psi_{\lambda^n} \|^2 \int_0^\infty \int_0^l \kappa(y,s)dyds \\ &= 2C \| u_{\lambda^n} \|_\infty \max\{\lambda^n\}. \end{aligned}$$

So we have $\lim_{n\to\infty} |T_{\lambda_n}| = 0$. Because u_{λ^n} is the principal eigenfunction of $A(\lambda^n)$ with the principal eigenvalue 0, we see that $\langle A(\lambda^n)\phi_{\lambda^n}, \phi_{\lambda^n} \rangle \leq 0$, which implies that

$$0 \leq \text{Re}(\mu_{\lambda_n}) \leq \text{Re}(T_{\lambda^n}) \leq |T_{\lambda^n}|, \tag{3.8}$$

and hence $\lim_{n\to\infty} \text{Re}(\mu_{\lambda_n}) = 0$. Similarly, we have

$$0 \leq \text{Im}(\mu_{\lambda_n}) \leq \text{Im}(T_{\lambda^n}) \leq |T_{\lambda^n}|, \tag{3.9}$$

which implies that $\lim_{n\to\infty} \text{Im}(\mu_{\lambda_n}) = 0$, where $\text{Re}(\mu_{\lambda_n})$ denotes the real part of μ_{λ_n} and $\text{Im}(\mu_{\lambda_n})$ denotes the imaginary part of μ_{λ_n}.

Since $|\langle A(\lambda^n)\phi_{\lambda^n},\phi_{\lambda^n}\rangle| \geq |\lambda_2(\lambda^n)|\cdot \parallel \phi_{\lambda^n} \parallel^2_{Y_C}$, where $\lambda_2(\lambda^n)$ is the second eigenvalue of $A(\lambda^n)$, we have

$$|\lambda_2(\lambda^n)|\cdot \parallel \phi_{\lambda^n} \parallel^2_{Y_C} \leq |T_{\lambda^n}| + |\mu_{\lambda^n}|. \tag{3.10}$$

And the continuity of the eigenvalues of $A(\lambda)$ with respect to λ implies that $\lim_{n\to\infty}\lambda_2(\lambda^n) = -\lambda_2+\lambda_0 < 0$, where $\lambda_2 = -4\lambda_0$ is the second eigenvalue of the operator $-\lambda_0\frac{\partial^2}{\partial x^2}$.

Noting that $\lim_{n\to\infty}|T_{\lambda^n}| = \lim_{n\to\infty}|\mu_{\lambda^n}| = 0$, we have from equation (2.21) that $\lim_{n\to\infty} \parallel \phi_{\lambda^n} \parallel_{Y_C}= 0$. Because of $\psi_{\lambda^n} = c_{\lambda^n}u_{\lambda^n} + \phi_{\lambda^n}$ and $\parallel \psi_{\lambda_n} \parallel_{L^2}= 1$, we can obtain

$$\lim_{n\to\infty}|c_{\lambda^n}|(\lambda^n-\lambda_0)\lim_{n\to\infty} \parallel \frac{u_{\lambda_n}}{\lambda^n-\lambda_0} \parallel_{Y_C}= 1$$

which implies that $\lim_{n\to\infty}|c_{\lambda^n}|(\lambda^n-\lambda_0) > 0$. We denote

$$\begin{aligned} Q_1 =& \lambda^n\langle\psi_{\lambda^n},\lambda^n u_{\lambda^n}h_1(x)\psi_{\lambda_n}\rangle,\\ Q_2 =& \lambda^n\langle\psi_{\lambda^n},u_{\lambda^n}h_2(x)\int_0^\infty\int_0^l\kappa(y,s)f(\psi_{\lambda_n}(y)e^{-\mu_{\lambda_n}s})dyds\rangle, \end{aligned}$$

and then $T_{\lambda^n} = Q_1+Q_2$. We first calculate that

$$\frac{Q_2}{\lambda^n-\lambda_0} \leq |c_{\lambda^n}|^2(\lambda^n-\lambda_0)^2\lambda^n\beta_1 + |c_{\lambda^n}|(\lambda^n-\lambda_0)\lambda^n\beta_2 + \overline{c_{\lambda^n}}(\lambda^n-\lambda_0)\lambda^n\beta_3 + \lambda^n\beta_4, \tag{3.11}$$

where

$$\begin{aligned} \beta_1 =& \int_0^l\int_0^\infty\int_0^l\kappa(y,s)e^{-\mu_{\lambda_n}s}h_2(x)\frac{u^2_{\lambda^n}(x)u_{\lambda^n}(y)}{(\lambda^n-\lambda_0)^3}dydsdx,\\ \beta_2 =& \int_0^l\int_0^\infty\int_0^l\kappa(y,s)e^{-\mu_{\lambda_n}s}h_2(x)\frac{\overline{\phi_{\lambda^n}}(y)u^2_{\lambda^n}(x)}{(\lambda^n-\lambda_0)^2}dydsdx,\\ \beta_3 =& \int_0^l\int_0^\infty\int_0^l\kappa(y,s)e^{-\mu_{\lambda_n}s}h_2(x)\frac{\phi_{\lambda^n}(x)u_{\lambda^n}(x)u_{\lambda^n}(y)}{(\lambda^n-\lambda_0)^2}dydsdx,\\ \beta_4 =& \int_0^l\int_0^\infty\int_0^l\kappa(y,s)e^{-\mu_{\lambda_n}s}h_2(x)\frac{\phi_{\lambda^n}(x)\phi_{\lambda^n}(y)u_{\lambda^n}(x)}{(\lambda^n-\lambda_0)}dydsdx. \end{aligned} \tag{3.12}$$

Since $\lim_{n\to\infty} \parallel \phi_{\lambda^n} \parallel_{Y_C=0}$, we have $\lim_{n\to\infty} \parallel \phi_{\lambda^n} \parallel_{L^1=0}$, which implies that each of $\beta_i(i=2,3,4)$ goes to zero as $n\to\infty$. Since μ_{λ^n} has nonnegative real parts with $\lim_{n\to\infty}|\mu_{\lambda^n}| = 0$, and $\lim_{n\to\infty}h_2(x) = b$ uniformly, we have

$$\lim_{n\to\infty}\beta_1 = b\alpha^3_{\lambda_0}\int_0^l\sin^3\frac{\pi x}{l}dx.$$

Then we have

$$\begin{aligned} \frac{Q_1}{\lambda^n-\lambda_0} \leq& |c_{\lambda^n}|^2(\lambda^n-\lambda_0)^2\lambda^n\int_0^l\frac{h_1(x)u^3_{\lambda_n}(x)}{(\lambda^n-\lambda_0)^3}dx + c_{\lambda^n}(\lambda^n-\lambda_0)\lambda^n\int_0^l\frac{h_1(x)\overline{\phi_{\lambda_n}}(x)u^2_{\lambda^n}(x)}{(\lambda^n-\lambda_0)^2}dx\\ &+\overline{c_{\lambda^n}}(\lambda^n-\lambda_0)\lambda^n\int_0^l\frac{h_1(x)\phi_{\lambda_n}(x)u^2_{\lambda^n}(x)}{(\lambda^n-\lambda_0)^2}dx + \lambda^n\int_0^l\frac{h_1(x)|\phi_{\lambda_n}(x)|^2u_{\lambda^n}(x)}{(\lambda^n-\lambda_0)}dx. \end{aligned} \tag{3.13}$$

Similarly, since $\lim_{n\to\infty} \parallel \phi_{\lambda^n} \parallel_{Y_C=0}$, we have that each of the last three terms of equation (2.24) goes to zero, and

$$\lim_{n\to\infty}\int_0^l\frac{h_1(x)u^3_{\lambda^n}(x)}{(\lambda^n-\lambda_0)^3}dx = a\alpha^3_{\lambda_0}\int_0^l\sin^3\frac{\pi x}{l}dx.$$

Therefore,

$$\lim_{n\to\infty}\frac{T_{\lambda_n}}{\lambda^n-\lambda_0} = \lambda_0\alpha^3_{\lambda_0}(a+b)\int_0^l\sin^3\frac{\pi x}{l}dx\cdot\lim_{n\to\infty}|c_{\lambda^n}|^2(\lambda^n-\lambda_0)^2 < 0.$$

So, for sufficiently large n, $Re(T_{\lambda^n}) < 0$, and consequently,

$$Re(\mu_{\lambda_n}) = \langle A(\lambda^n)\phi_{\lambda^n},\phi_{\lambda^n}\rangle + Re(T_{\lambda^n}) < 0 \tag{3.14}$$

which is a contradiction with $Re(T_{\lambda^n}) \geq 0$ for $n \geq 1$. Therefore, we have

$$S(\lambda) \subset \{x+iy : x,y\in R, X<0\}.$$

The proof is completed. ■

4 Conclusion

From the Theorem 2, the eigenvalues of the linearization system (3.1) have negative real parts. So we can easily get the local stability of positive steady state u_λ.

Theorem 3 *Suppose that the constants a, b and $f'(0)$ satisfy equation (2.3), then there exists $\lambda_1 > \lambda_0$, where $\lambda_1 \leq \lambda^*$, such that for any $\lambda \in (\lambda_0, \lambda_1]$, the positive steady state u_λ of equation (2.1) is locally asymptotically stable.*

In this paper, by using the implicit function theorem we establish the existence of a positive spatially nonhomogeneous equilibrium for the model. Meanwhile, we prove that the positive spatially nonhomogeneous equilibrium can bifurcate from the trivial equilibrium. Under some conditions it is founded that for the given spatiotemporal delay, the bifurcated positive spatially nonhomogeneous equilibrium is stable, and the Hopf bifurcation cannot occur. From the above theorems we know that when $\lambda > \lambda_0$ and close to λ_0, a delay τ on a domain can make the unique spatially nonhomogeneous positive equilibrium of (1.1). How to overcome technical problems that prevent a full analysis of the delay on the whole domain deserving further investigation.

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