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The Singular Integral Related to The Tricomi Type Operators

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Abstract: In this paper the singular integral related to the Tricomi type operators are studied. We use Stein's [17] "cancelation" criterion and weighted inequalities for singular integrals.

Keywords: singular integral; Tricomi type operator; A_p weights

1 Introduction and main results

Consider the Tricomi type operator

$$Tu = y^p u_{xx} + u_{yy} \tag{1}$$

in the upper plane $\mathbb{R}^2_+ = \{z = (x, y) \in \mathbb{R}^2, y > 0\}$ with $p > 0$ a real number. Results that in $\mathbb{R}^2_+$ the operator has a solution $E(z, z_0)$, having a logarithmic singularity at point $z_0 = (x_0, y_0) \in \mathbb{R}^2_+$ from [3, 19]. Here

$$E = E(z, z_0) = (s_0 s)^{-\alpha} A_\alpha(\zeta), \tag{2}$$

where $\alpha = \frac{p}{2(p+2)}$,

$$A_\alpha(\zeta) = c_\alpha(-\zeta)^\alpha F(\alpha, \alpha, 2\alpha; \zeta), \tag{3}$$

$$\zeta = -\frac{4s_0 s}{r^2}, \quad r^2 = (x - x_0)^2 + (s - s_0)^2, \tag{4}$$

$s = \frac{2}{p+2} y^{\frac{p+2}{2}}$ and $s_0 = \frac{2}{p+2} y_0^{\frac{p+2}{2}}$, and the constant

$$c_\alpha = \frac{1}{4} \cdot \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}.$$

E can be regarded as solutions of (1) with Neumann boundary conditions on the boundary $y = 0$ of $\mathbb{R}^2_+$. Actually, explicit calculation yields

$$\lim_{y\to 0} E = c_\alpha[(x - x_0)^2 + s_0^2]^{-2\alpha}, \quad \lim_{y\to 0} \partial_y E = \lim_{y\to 0} \partial_{yy} E = 0.$$

Let $a > 0$ be real number and $\chi(y) \in C^\infty(\mathbb{R})$ such that $\chi(y) = 1$ if $y \geq \frac{a}{2}$ and $\chi(y) = 0$ if $y < \frac{a}{4}$. One can extends $E(z, z_0)$ to $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{z = z_0\}$ by setting

$$g(z, z_0) = E(z, z_0)\chi(y)\chi(y_0). \tag{5}$$

Notice that $g(z, z_0)$ vanishes if one of y and y_0 is less than $\frac{a}{4}$.

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Write

$$u(z) = \int_{\mathbb{R}^2} g(z, z_0) f(z_0) dz_0, \tag{6}$$

and, for $i = 1, 2$

$$f_{ii}(z) = \int_{\mathbb{R}^2} X_i X_i g(z, z_0) f(z_0) dz_0 \tag{7}$$

with

$$X_1 = |y|^{\frac{p}{2}} \partial_x, \quad X_2 = \partial_y.$$

The main result of this paper is

Theorem 1 *Let $0 < p < 2$. Suppose that $f(z)$ has compact support above the line $y > a$ and is integrable. Then, the singular integrals f_{ii} exist and their L^2 norm are controlled by that of f, that is*

$$\|f_{ii}\| \leq C \|f\| \tag{8}$$

for $i = 1, 2$. The constant C is independent of f.

Remark 2 *If $p = 0$, T is the Laplace operator, Calderón and Zygmund [8] obtained the estimates to (8); If $p > 1$ and Ω is a smooth bounded domain of $\mathbb{R}^n_+$, Baouendi [1] obtained the similar estimates near the degenerate line; See also Rothschild and Stein [14] for sharp estimates by fundamental solution on nilpotent Lie group. Ruan, Witt and Yin [15, 16] studied the singular integral (8) for (1) when $p > 0$ with bounded domain.*

2 Standard kernels

Recall $X_1 = |y|^{\frac{p}{2}} \partial_x$, $X_2 = \partial_y$. Let $g(z, z_0),\ z, z_0 \in \mathbb{R}^2$ be defined in the introduction. Set

$$K_{ii}(w, w_0) = |y|^{-p} |s_0 s|^{\alpha} X_i X_i g(z, z_0) \tag{9}$$

for $i = 1, 2$, $w = (x, s)$, $w_0 = (x_0, s_0)$ with

$$s = \frac{2}{p+2} |y|^{\frac{p}{2}} y, \quad s_0 = \frac{2}{p+2} |y_0|^{\frac{p}{2}} y_0. \tag{10}$$

In (4), ζ and r^2 is restricted to $s_0, s > 0$, now we extend these depended variables to $s_0, s \in \mathbb{R}$ by set

$$\zeta = -\frac{4|s_0 s|}{r^2}, \quad r^2 = (x - x_0)^2 + (s - s_0)^2. \tag{11}$$

Then we have

$$g(z, z_0) = |s_0 s|^{-\alpha} A_\alpha(\zeta) \chi_1(s) \chi_1(s_0) \tag{12}$$

with $A_\alpha(\zeta)$ defined in (3) and $\chi_1(s) = \chi(y)$.

Now

$$K_{11}(w, w_0) = \chi_1(s) \chi_1(s_0) \partial_{xx} A_\alpha(\zeta). \tag{13}$$

By (1), one has

$$K_{22} = -K_{11} \tag{14}$$

if $y, y_0 \geq \frac{a}{2}$.

Lemma 3 *We have the asymptotic behaviors*

$$\partial_{xx} A_\alpha(\zeta) = \frac{(x-x_0)^2 - (s-s_0)^2}{2r^4} + D_\alpha, \tag{15}$$

as $\zeta \to -\infty$. *Here*

$$D_\alpha = \frac{l_0}{2}(\log(-\zeta) - 2(x-x_0)^2 r^{-2} + n_0 - 1) r^{-2}\zeta^{-1} + O(\log(-\zeta)\cdot r^{-2}\zeta^{-2}),$$

and

$$\partial_{xx} A_\alpha(\zeta) = \frac{(x-x_0)^2 - (s-s_0)^2}{2r^4} + o(1), \tag{16}$$

as $\zeta \to 0$.

Proof.

Set

$$F_k = F_{k,\alpha} = F(\alpha+k, \alpha+k, 2\alpha+k; \zeta)$$

for $k = 0, 1, 2$. Using the formula

$$\frac{d}{dz}F(a,b,c;z) = \frac{ab}{c}F(a+1, b+1, c+1; z),$$

we get

$$\partial_x F_0 = \frac{\alpha}{2} F_1 \zeta_x, \quad \partial_{xx} F_0 = \frac{\alpha}{2} F_1 \zeta_{xx} + \frac{\alpha(\alpha+1)^2}{2(2\alpha+1)} F_2 \cdot (\zeta_x)^2.$$

By definition of ζ we have

$$\zeta_x = -2(x-x_0) r^{-2}\zeta, \quad \zeta_{xx} = -2[(s-s_0)^2 - 3(x-x_0)^2] r^{-4}\zeta.$$

Then

$$\partial_{xx}(-\zeta)^\alpha \cdot F_0 = 2\alpha\left[(2\alpha+1)(x-x_0)^2 - (s-s_0)^2\right] r^{-4}\cdot(-\zeta)^\alpha F_0, \tag{17}$$

$$2\partial_x(-\zeta)^\alpha \partial_x \cdot F_0 = -4\alpha^2 (x-x_0)^2 r^{-4}\cdot(-\zeta)^{\alpha+1} F_1, \tag{18}$$

and

$$\begin{aligned}(-\zeta)^\alpha \partial_{xx} F_0 = \alpha[(s-s_0)^2 - 3(x-x_0)^2] r^{-4}\cdot(-\zeta)^{\alpha+1} F_1 \\ + \frac{2\alpha(\alpha+1)^2}{2\alpha+1}(x-x_0)^2 r^{-4}\cdot(-\zeta)^{\alpha+2} F_2.\end{aligned} \tag{19}$$

By formula (36), one has

$$(-\zeta)^{\alpha+k} F_k = \log(-\zeta) U_k(\zeta) + V_k(\zeta) \tag{20}$$

with

$$U_k(\zeta) = u_k[1 + l_k\zeta^{-1} + O(\zeta^{-2})], \quad V_k(\zeta) = u_k[m_k + l_k n_k \zeta^{-1} + O(\zeta^{-2})],$$

as $\zeta \to -\infty$. Here are quantities depending α and k:

$$u_k = \frac{\Gamma(2\alpha+k)}{\Gamma(\alpha+k)\Gamma(\alpha)}, \quad l_k = (1-\alpha)(\alpha+k),$$

$$m_k = h_0(\alpha+k, 2\alpha+k) = 2\Psi(1) - \Psi(\alpha+k) - \Psi(\alpha),$$

and

$$n_k = h_1(\alpha + k, 2\alpha + k) = 2\Psi(2) - \Psi(\alpha + k + 1) - \Psi(\alpha - 1).$$

By formula $\Gamma(1 + z) = z\Gamma(z)$, we have

$$u_0 = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}, \quad u_1 = 2u_0, \quad u_2 = \frac{2(2\alpha + 1)}{\alpha + 1} u_0. \tag{21}$$

Using formula $\Psi(1 + z) = \Psi(z) + \frac{1}{z}$, we have

$$m_0 = 2\Psi(1) - 2\Psi(\alpha), \quad m_1 = m_0 - \frac{1}{\alpha}, \quad m_2 = m_0 - \frac{2\alpha + 1}{\alpha(\alpha + 1)}.$$

Set $v_k = u_k m_k$. We have

$$v_0 = u_0 m_0, \quad v_1 = 2v_0 - \frac{2}{\alpha} u_0, \quad v_2 = \frac{2(2\alpha + 1)}{\alpha + 1} v_0 - \frac{2(2\alpha + 1)^2}{\alpha(\alpha + 1)^2} u_0. \tag{22}$$

Using formula $\Psi(1 + z) = \Psi(z) + \frac{1}{z}$, we have

$$n_0 = 2\Psi(2) - 2\Psi(\alpha) - l_0^{-1}, \quad n_1 = n_0 - \frac{1}{\alpha + 1}, \quad n_2 = n_0 - \frac{2\alpha + 3}{(\alpha + 1)(\alpha + 2)}.$$

Set $w_k = u_k l_k n_k$. Then $w_0 = u_0 l_0 n_0$, $w_1 = 2w_0 + 2(1 - \alpha)u_0(n_0 - 1)$,

$$w_2 = \frac{2(2\alpha + 1)}{\alpha + 1} w_0 + \frac{2(2\alpha + 1)(1 - \alpha)}{\alpha + 1} u_0 [2n_0 - \frac{2\alpha + 3}{\alpha + 1}]. \tag{23}$$

Note $4c_\alpha = \frac{1}{u_0}$ and $A_\alpha(\zeta) = c_\alpha(-\zeta)^\alpha F_0$. By (20)

$$4\partial_{xx} A_\alpha(\zeta) = \log(-\zeta) \cdot P + Q \tag{24}$$

with

$$P = (x - x_0)^2 r^{-4}(c_1 + c_2\zeta^{-1}) + (s - s_0)^2 r^{-4}(c_3 + c_4\zeta^{-1}) + O(r^{-2}\zeta^{-2})$$

and

$$Q = (x - x_0)^2 r^{-4}(c_5 + c_6\zeta^{-1}) + (s - s_0)^2 r^{-4}(c_7 + c_8\zeta^{-1}) + O(r^{-2}\zeta^{-2}),$$

as $\zeta \to -\infty$. Using (21),

$$c_1 = \alpha(2\alpha + 1) - 8\alpha^2 - 6\alpha + 4\alpha(\alpha + 1) = 0$$

and

$$c_3 = -2\alpha + 2\alpha = 0.$$

Similarly, we get $c_2 = c_4 = 2l_0$. Then

$$P = 2l_0 r^{-2}\zeta^{-1} + O(r^{-2}\zeta^{-2}) \quad \text{as } \zeta \to \infty. \tag{25}$$

Using (22),

$$c_5 = (2\alpha(2\alpha + 1) - 8\alpha^2 - 6\alpha + 4\alpha(\alpha + 1))m_0 + 8\alpha + 6 - 4(2\alpha + 1) = 2$$

and

$$c_7 = (-2\alpha + 2\alpha)m_0 - 2 = -2.$$

Similarly, we get $c_6 = 2l_0(n_0 - 3)$, $c_8 = 2l_0(n_0 - 1)$. Then

$$Q = 2 \cdot \frac{(x-x_0)^2 - (s-s_0)^2}{r^4} - 4l_0(x-x_0)^2 r^{-4}\zeta^{-1} + 2l_0(n_0-1)r^{-2}\zeta^{-1} + O(r^{-2}\zeta^{-2}) \tag{26}$$

as $\zeta \to \infty$. l_0 takes the same value $\alpha(1-\alpha)$. Combining (25) and (26) yield

$$\partial_{xx}A_\alpha(\zeta) = \frac{(x-x_0)^2 - (s-s_0)^2}{2r^4} + D_\alpha \tag{27}$$

as $\zeta \to -\infty$. So (15) is valid.

By (18), we get $|\partial_{xx}A_\alpha|,\ |\partial_{xx}A_{1-\alpha}| \leq C\frac{1}{r^{2+\alpha}}$ as $r \to \infty$.

Note that $(-\zeta)^{\alpha+k}F_k$ tends to 0 as $\zeta \to 0$, thus the estimate of (16) is verified. ■

Definition 1 *A function $K(w, w_0)$, defined for $w \neq w_0$, $w, w_0 \in \mathbb{R}^2$ is called a "Standard kernel" if it is assumed to satisfy the basic inequalities: For some γ, $0 < \gamma < 1$, one has*

$$|K(w, w_0)| \leq C|w - w_0|^{-2} \tag{28}$$

and

$$|K(w, w_0) - K(w, w_1)| + |K(w_0, w) - K(w_1, w)| \leq C\frac{|w_0 - w_1|^\gamma}{|w - w_0|^{2+\gamma}}, \tag{29}$$

if $|w_0 - w_1| < |w - w_0|/2$.

Proposition 4 *K_{11}, K_{22} are standard kernels.*

Proof. By Lemma 3, K_{11} is a standard kernel. The proof of K_{22} follows from (14). ■

Definition 2 *A singular integral operator T associated to a standard kernel $K(w, w_0)$ is a bounded linear operator from the space $\mathcal{D} = C_c^\infty(\mathbb{R}^2)$ of test functions to its dual $\mathcal{D}'$ (the space of distributions), such that if $f \in \mathcal{D}$, then, outside the support of f, the distribution Tf agrees with the function*

$$(Tf)(w) = \int_{\mathbb{R}^2} K(w, w_0) f(w_0) dw_0. \tag{30}$$

Given a standard kernel $K(w, w_0)$, what "cancelation" conditions must be imposed on K so that there exists a bounded operator $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ having kernel K in the sense of (30). Stein [17] introduces the quantities

$$I_{\varepsilon,N}(w) = \int_{\varepsilon<|w-w_0|<N} K(w, w_0) dw_0$$

and that for the adjoint kernel $\bar{K}(w_0, w)$

$$I^*_{\varepsilon,N}(w_0) = \int_{\varepsilon<|w-w_0|<N} \bar{K}(w, w_0) dw.$$

Theorem 5 *Stein [17], P306, Suppose that $K(w, w_0)$ is a standard kernel. Then there exists a bounded operator T : $L^2 \to L^2$ so that (30) holds if and only if there is an $A > 0$ such that*

$$\int_{|w-w_1|<N} |I_{\varepsilon,N}(w)|^2 dw \leq A \cdot N^2 \tag{31}$$

for all ε, N, and w_1, and a similar condition for $I^*_{\varepsilon,N}(w_0)$.

David and Journé: weakly bounded, $T1, T^*1 \in$BMO, homogenous space.

Proposition 6 *Let the functions $K_{11}(w, w_0)$ and $K_{22}(w, w_0)$ be defined in (9). Then there exist bounded operators T_{11} and T_{22} from $L^2(\mathbb{R}^2)$ to itself having kernels K_{11} and K_{22} in the sense of (30) respectively.*

Proof. Clearly, we have $I_{\varepsilon,N}(w) = I^*_{\varepsilon,N}(w)$. By Proposition 4, K_{11}, K_{22} are standard kernels. By Theorem 5, only the "cancelation" conditions (31) need be verified.

Without loss of generality, we assume that $a = 1$. By Lemma 3, $K_{11}(w, w_0)$ satisfies the properties (31) and T_{11} is a bounded operators from $L^2(\mathbb{R}^2)$ to itself having the kernel K_{11}. The proof of K_{22} is similar. ■

Definition 3 *A nonnegative locally integrable function ω is called in the class (A_2) if it satisfies the inequality*

$$\frac{1}{|Q|}\int_Q \omega(w)dw \cdot \frac{1}{|Q|}\int_Q \omega(w)^{-1}dw < C < \infty \tag{32}$$

for all cubes Q in $\mathbb{R}^2$, here $|Q|$ is the Lebesgue measure of the set $Q \subset \mathbb{R}^2$.

Theorem 7 *Stein [17], P221, Let T be a bounded operator from $L^2(\mathbb{R}^2)$ to itself that is associated to a standard kernel $K(w, w_0)$ in the sense of (30), and $\omega \in (A_2)$. Then for all $f \in C_0^\infty(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} |Tf(w)|^2\omega(w)dw \le A\int_{\mathbb{R}^2} |f(w)|^2\omega(w)dw. \tag{33}$$

Remark 8 *The results for translation-invariant operator is given by Coifman an Fefferman, and are extended to non-translation-invariant operator by Stein in his book.*

Proof of Theorem 1

By variables change (10), one has

$$\|f_{11}\|^2 = \bar{p}^4\int_{\mathbb{R}^2} |T_{11}h(w)|^2|s|^{4\alpha}dw \tag{34}$$

with $w = (x, s)$, and

$$(T_{11}h)(w) = \int_{\mathbb{R}^2} K_{11}(w, w_0)h(w_0)dw_0,$$

where $w_0 = (x_0, s_0)$, and

$$h(w_0) = \bar{p}f(z_0)|s_0|^{-3\alpha}, \quad K_{11}(w, w_0) = \chi_1(s)\chi_1(s_0)\partial_{xx}A_\alpha(\zeta).$$

Here $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^2)$, $\bar{p}$ is some positive constant related only to p.

By Proposition 6, the above operator T_{11} is a bounded linear operator from $L^2(\mathbb{R}^2)$ to itself associated to the standard kernel $K_{11}(w, w_0)$ in the sense of (30). Direct calculation imply that $\omega(w) = |s|^{4\alpha}$ is in the class (A_2) if $-1 < 4\alpha < 1$, that is $-2 < p < 2$. By Theorem 7, the integral of the right side of (34) is controlled by

$$\int_{\mathbb{R}^2} |h(w)|^2|s|^{4\alpha}dw = \int_{\mathbb{R}^2} |f(z)|^2dxdy = \|f\|^2,$$

that is

$$\|f_{11}\|^2 \le C\|f\|^2 \tag{35}$$

for some constant C depending only on p and a. By (14), similar arguments lead to the estimate for f_{22}.

3 Appendix

Denote by

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Let a, b, and c be complex numbers, $c \neq 0, -1, -2, \ldots$. The power series

$$F(a,b,c;\zeta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \zeta^n$$

is called *hypergeometric series*. It is a solution of the hypergeometric equation

$$\zeta(1-\zeta)\frac{d^2u}{d\zeta^2} + \{c-(a+b+1)\zeta\}\frac{du}{d\zeta} - abu = 0.$$

The series is absolutely convergent for all numbers $|\zeta| < 1$.

Let $|\arg(-\zeta)| < \pi$ be the domain of the complex plane $\mathbb{C}$ minus the positive real axis. Then Barnes's contour integral extends the hypergeometric series $F(a,b,c;\zeta)$ to a single-valued analytic function of ζ in the region $|\arg(-\zeta)| < \pi$, which still denoted by $F(a,b,c;\zeta)$. In particular, if $a = b$, then

$$F(a,a,c;\zeta) = (-\zeta)^{-a}[\log(-\zeta)U(\zeta) + V(\zeta)], \tag{36}$$

where

$$U(\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{j=0}^{\infty} \frac{(a)_j(1-c+a)_j}{j!j!}\zeta^{-j}, \tag{37}$$

and

$$V(\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{j=0}^{\infty} \frac{(a)_j(1-c+a)_j}{j!j!} h_j \zeta^{-j}, \tag{38}$$

here

$$h_j = h_j(a,c) = 2\Psi(1+j) - \Psi(a+j) - \Psi(c-a-j)$$

and

$$\Psi(z) = \Gamma'(z)/\Gamma(z).$$

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