Global Existence of Piecewise $\varepsilon$–Regular Solutions for a Class of Abstract Parabolic Equations

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(Received 18 March 2013, accepted 2 July 2013)

Abstract: We deal with a class of abstract parabolic equations with critical nonlinearities. By adding $\mathbb{w}_\varepsilon$–continuous assumption on the nonlinearities, we establish the relation between the global existence and $Z^1$–boundedness for the piecewise $\varepsilon$–regular solutions of the equations. As an application, we show that, every critical nonlinear term attached to the strongly damped wave equation $u_{tt} + \eta(-\Delta)^\theta u_t + (-\Delta) u = f(u)$ ($\theta \in [\frac{1}{2}, 1]$) exhibits exactly the $\mathbb{w}_\varepsilon$–continuity in the spaces dual $(Y, Y_\theta)$. Consequently each piecewise $\varepsilon$–regular solution of the wave equation exists universally or it blows up in the energy space $Y$ in finite time.

Keywords: abstract parabolic equation; sectorial operator; fractional power; piecewise $\varepsilon$–regular solution; global existence

1 Introduction

In this paper, we study the Cauchy problem of the following abstract parabolic equation

$$\frac{dz}{dt} + Pz = F(z), \quad t > 0, \quad z(0) = z_0$$

(1.1)

Here, $Z$ is a Banach space, and $P : D(P) \subseteq Z \to Z$ is a sectorial operator. For each $\alpha > 0$, denote by $P^{\alpha}$ the fractional power of $P$ and $Z^\alpha = D(P^{\alpha})$ the fractional power space endowed with the norm $\| \cdot \|_{Z^\alpha} = \| P^{\alpha} \cdot \|_Z$.

An important task attached to equation (1.1) is to investigate the existence, regularity and asymptotic actions of its strong solutions. If $F : Z^1 \to Z^\alpha$ is locally Lipschitz for some $\alpha \in (0, 1]$, then for each $z_0 \in Z^1$, there is a $\tau_0 > 0$ (perhaps $\tau_0 = +\infty$) and a unique function $z \in C([0, \tau_0], Z^1) \cap C^1((0, \tau_0), Z)$, satisfying equation (1.1) and the initial condition (1.2). The interval $[0, \tau_0)$ is called maximal if the solution $z(\cdot)$ could not extended beyond it (see [1, 2] for reference).

If the range $\mathcal{R}(F)$ could not be contained in $Z^\alpha$ for any $\alpha \in (0, 1]$, then the problem mentioned above become much sophisticated. By introducing the notations of $\varepsilon$–regular map and $\varepsilon$–regular solution, Arrieta-Carvalho in [3] investigated the local existence and differentiability of $\varepsilon$–regular solutions for (1.1) with $Z^1$ initial value and $\varepsilon$–regular nonlinearity $F$. As applications, Arrieta-Carvalho in [3] and Arrieta-Carvalho-Bernal in [4] studied the $L^q$ and $W^{1,q}$–regular solutions of the heat equations with Dirichlet and nonlinear boundary conditions respectively. Carvalho-Cholewa in [5] studied the $H^1 \times L^2$–regular solutions of the strongly damped wave equations with fractional powers of the negative Laplacian, and lately Carvalho-Cholewa-Dlotko in [6] extended the results obtained in [5] to a more general case. In [8], Carvalho-Cholewa considered global existence of $\varepsilon$–regular solutions. They gave a sufficient and necessary condition, and introduced the notion of piecewise $\varepsilon$–regular solution in case that an $\varepsilon$–regular solution blows up in finite time.

This paper focuses on the global existence of the piecewise $\varepsilon$–regular solutions, which is posed in [8]. We will give a sufficient condition for this problem in Section 2 together with a brief proof. As an application, in Section 3, we will show that, every critical nonlinear term attached to the strongly damped wave equation exhibits exactly the $\mathbb{w}_\varepsilon$–continuity in

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IJSNS.2013.10.15/749
the spaces dual \((Y, Y_{0,1})\). Consequently each piecewise \(\varepsilon\)-regular solution of the wave equation exists universally once its \(Y\)-norm is proved to be bounded on the maximal existing interval. Tools used here are theory of sectorial operators and analytic semigroups together with nonlinear analysis.

2 Global existence of piecewise \(\varepsilon\)– regular solutions

Firstly, let us review briefly the definitions and important results mentioned above for later use. Let \(\rho > 1\), and \(\varepsilon \in (0, 1/\rho)\).

Definition 2.1 A nonlinear map \(F\) is said to be \(\varepsilon\)-regular associated with the pair \((X, X^1)\), if there is a number \(\gamma \in [\rho\varepsilon, 1)\) and a constant \(C > 0\) satisfying

\[
\|F(z_1) - F(z_2)\|_{Z^\gamma} \leq C\|z_1 - z_2\|_{Z^{1+\varepsilon}} \left(1 + \|z_1\|_{Z^{1+\varepsilon}}^{-\rho\varepsilon} + \|z_2\|_{Z^{1+\varepsilon}}^{-\rho\varepsilon}\right).
\]

(2.1)

Definition 2.2 A function \(z \in C([0, \tau_0], Z^1)\) is called the \(\varepsilon\)-regular solution of problem (1.1)+(1.2), if \(z \in C((0, \tau_0), Z^{1+\varepsilon}) \cap C^1((0, \tau_0), Z)\) verifying (1.1) in \(Z\) together with the initial condition (1.2) verified.

Proposition 2.3 For the \(\varepsilon\)-regular nonlinearity \(F\) and the element \(\tilde{z} \in Z^1\), there are correspondingly two numbers \(\tau, T \geq 0\), s.t. for each \(z_0 \in B_{Z^1}(\tilde{z}, \tau)\), problem (1.1)+(1.2) permits a unique \(\varepsilon\)-regular solution \(z(t, z_0)\) defined on \([0, T]\) satisfying \(z \in C([0, T], Z^1) \cap C^1((0, T), Z^{1+\varepsilon})\). If \(\gamma > \rho\varepsilon\), then the number \(T\) depends not on \(\tau\) but on the radius \(\tau\). This in case, \(z\) survives globally or its \(Z^1\)-norm blows up in finite time (see [3] or Ch. 2, [7]).

The last conclusion in Prop. 2.3 does not keep right any more in the critical case \(\gamma = \rho\varepsilon\). Any other assumptions are needed then. In [3], the authors provided a sufficient one: \(\sup_{z \in \Omega_{\varepsilon}} \|z(t)\|_{Z^\sigma} < +\infty\) for some \(\sigma > 1\). And [8] proved that if \(F\) is an almost critical \(\varepsilon\)-regular map, then \(z\) exists globally if its \(Z^{1+\varepsilon}\)-norm remains bounded all the time. The authors also found a sufficient and necessary condition for the universal existence of \(z\), that is

\(H\): for any \(\tau \in (0, \tau_0]\), solution \(z(\cdot)\) is uniformly continuous on \([0, \tau]\) in \(Z^1\).

\(H(2)\): the energy estimate \(\|z(t, z_0)\|_{Z^1} \leq g(z_0)\) holds for all \(t \geq 0\), where \(g : Z^1 \to \mathbb{R}^+\) is bounded on bounded subsets of \(Z^1\).

\(H(3)\): if \(\tau_0 < 0\), then there is a Banach space \(Y\) with \(Z^1 \hookrightarrow Y\), s.t. the \(\varepsilon\)-regular solution \(z(\cdot, z_0)\) is uniformly continuous on \([0, \tau_0]\) according to the \(Y\)-norm, and proved.

Proposition 2.4 If all the hypotheses listed above but \(H\) come true, then every \(\varepsilon\)-regular solution \(z(\cdot)\) of (1.1) beginning in \(Z^1\) can be extended beyond its maximal interval of existence \([0, \tau_0]\) as a piecewise \(\varepsilon\)-regular solution. More precisely, there is a \(\tau > \tau_0\) (perhaps \(\tau = +\infty\)) and an increasing sequence \(\{\tau_n\} \subseteq [\tau_0, \tau), s.t.

- \(z \in C([0, T], Y) \cap WCB([0, T], Z^1)\) for any \(T \in (\tau_0, \tau]\),
- on each interval \([\tau_{i-1}, \tau_i]\), \(z\) is the \(\varepsilon\)-regular solution of (1.1),
- \(\lim_{\tau_i \to \tau_i^+} \|z\|_{Z^{1+\varepsilon}} = +\infty\),
- \(\tau = +\infty\) or \(\lim_{i \to +\infty} \tau_i = \tau\).

Now, we begin to deal with the global existence of the piecewise \(\varepsilon\)-regular solution \(z\). Firstly, notice that on each subinterval \([\tau_{i-1}, \tau_i]\), \(z\) satisfies the following integral equation

\[
z(t) = e^{-(t-t_{i-1})P}z(t_{i-1}) + \int_{t_{i-1}}^{t} e^{-(t-s)P}F(z(s))ds, \quad t \in [\tau_{i-1}, \tau_i).
\]

(2.2)

This equality can help us to extend \(z\) in \(Z^{1+\varepsilon}\) beyond the singular points \(\tau_i\), \(i = 1, 2, \cdots\). But we could not realize the idea until now, since the \(Z^{1+\varepsilon}\)–norm of \(z(t)\) is unbounded about each \(\tau_i\), and consequently \(F(z(\cdot))\) could not be defined at \(\tau_i\). In order to overcome the difficulty, we have to lay some extra assumptions on \(F\). A natural one is that

\(H(4)\): \(F : Z^1 \to Z\) is bounded and \(w\)–continuous, i.e. for any sequence \(\{z_n\}\) convergent to \(z\) weakly in \(Z^1\), the corresponding sequence \(\{F(z_n)\}\) converges to \(F(z)\) in \(Z\) weakly.
Theorem 2.5 Under conditions $H(1) - H(4)$ with $Y \hookrightarrow Z$, every piecewise $\varepsilon$-regular solution $z(\cdot)$ of Eq. (1.1) with $z(0) \in Z^1$ exists globally.

Proof. We will prove the theorem by contradiction. Suppose $\tau < +\infty$, then the increasing sequence of singular points $\{\tau_i\}$ is convergent to $\tau$. In light of Prop. 2.4, $z$ is the $\varepsilon$-regular solution on each interval $[\tau_{i-1}, \tau_i)$, and for each $T \in (\tau_0, \tau)$, $z \in C([0, T], Y) \cap WCB([0, T], Z^1)$. Notice that $Y \hookrightarrow Z$, we have $z \in C([0, T], Z)$. Consider the multiple function $\mathcal{F}(z(\cdot))$, it is $w-$continuous and bounded on $[0, T]$ in the space $Z$ due to $H(4)$. Hence, the function $e^{-(t-s)P}\mathcal{F}(z(s))$ is also $w-$continuous and bounded on the area $\{(s, t) : 0 \leq s \leq t \leq T\}$. As an immediate result, for each $t \in [0, T]$, we have that $e^{-(t-s)P}\mathcal{F}(z(\cdot))$ is strongly measurable on $[0, t]$. Therefore the Bochner integration $\mathcal{H}(t) = \int_0^t e^{-(t-s)P}\mathcal{F}(z(s))ds$ is well defined and $w-$continuous on $[0, T]$ in the space $Z$.

Now, we turn to equality (2.2). Let $t \rightarrow \tau^{-}$, using the weak continuity of the both sides in $Z$, we can find that (2.2) is also verified at $\tau$, which means the interval on which (2.2) holds can be connected across the singular points. Furthermore, using $H(2)$, we have $\sup_{t \in (0, \tau)} \|z(t)\|_{Z^1} < +\infty$, thus $z \in C([0, \tau], Z) \cap WCB([0, \tau], Z^1)$. On the other hand, since $e^{-(\tau-s)P}\mathcal{F}(z(\cdot))$ is right-continuous weakly on $[0, \tau)$, it is strongly measurable on $[0, \tau)$, hence by the boundedness of $\mathcal{F}$, we can deduce that $\mathcal{H}(\tau)$ is well defined, which means, the function $\mathcal{H}(\cdot)$ can be extended to the whole closed interval $[0, \tau]$ with the boundedness in $Z$ preserved.

In the next step, we will show the $w-$continuity of $\mathcal{H}(\cdot)$ at $\tau$ in $Z$. For this purpose, take an increasing sequence $\{t_n\} \subseteq [0, \tau)$ convergent to $\tau$, and consider the difference $\mathcal{H}(\tau) - \mathcal{H}(t_n)$. Using the standard splitting, we have

$$
\mathcal{H}(\tau) - \mathcal{H}(t_n) = (e^{-(\tau-t_n)P} - I)\mathcal{H}(t_n) + \int_{t_n}^\tau e^{-(\tau-s)P}\mathcal{F}(z(s))ds = I_n + J_n,
$$

Taking any $\vartheta \in (Z)^*$, we get

$$
|\langle I_n, \vartheta \rangle| = |\langle \mathcal{H}(t_n), (e^{-(\tau-t_n)P^*} - I)\vartheta \rangle| \leq \sup_{t \in (0, \tau)} \|\mathcal{H}(t)\|_Z \cdot \|(e^{-(\tau-t_n)P^*} - I)\vartheta\|_{(Z)^*},
$$

where $P^*$ is the adjoint operator of $P$ and correspondingly $e^{-tP^*}$ is the dual semigroup of $e^{-tP}$. According to the strong continuity of $e^{-tP^*}$ and uniform boundedness of $\|\mathcal{F}(\cdot)\|_Z$, we have $\langle I_n, \vartheta \rangle \rightarrow 0$ as $n \rightarrow +\infty$. And by the absolute continuity of Bochner integrations, we have $J_n \rightarrow 0$ as $n \rightarrow +\infty$. Thus the weak continuity of $\mathcal{H}(\cdot)$ at $\tau$ has been reached.

Define $z(\tau) = e^{-\tau P}z(0) + H(\tau)$, we then obtain $z \in WC([0, \tau], Z) \cap B([0, \tau], Z^1)$). Due to the compact imbedding $Z^1 \hookrightarrow Z$, we have that for each sequence $\{t_n\}$ convergent to $\tau$, $\{z(t_n)\}$ converges weakly in $Z^1$, and strongly in $Z$, and the weak cluster $\omega_w(Z, \{z(t_n)\}) = \{z(\tau)\}$, which means $z \in WC([0, \tau], Z^1) \cap C([0, \tau], Z)$. Finally following the same process as in [8], we can prove that the piecewise $\varepsilon-$regular solution $z$ can be extended beyond $\tau$, which contradicts the maximaly of $[0, \tau]$. Thus the theorem has been proved.

Remark 2.6 From the last part of the above proof, we can find that condition $H(3)$ can be derived from $H(2)$ and $H(4)$ with $Y = Z$. In this sense, Thm. 2.5 also holds under conditions $H(1)$, $H(2)$, and $H(4)$.

Remark 2.7 We can also deduce from the above proof that the weak derivative of $z$ exists for all $t \geq 0$. In fact, for each $h > 0$, consider the quotient $\frac{1}{h}(\mathcal{H}(t+h) - \mathcal{H}(t))$. By the standard splitting

$$
\frac{1}{h}(\mathcal{H}(t+h) - \mathcal{H}(t)) = e^{-hP} - I + \int_0^t e^{-(t-s)P}\mathcal{F}(z(s))ds,
$$

again, we can easily deduce that the weak right derivative $\mathcal{H}_r(\cdot)$ exists in $Z$ and equals $PH(\cdot) + \mathcal{F}(z(t))$. Combining with the $w-$continuity of $P\mathcal{H}(\cdot)$ and $\mathcal{F}(\cdot)$, we conclude that the weak derivative $\mathcal{H}_r(\cdot)$ and hence $z'(\cdot)$ also exist.

Remark 2.8 Theorem 2.5 gives a positive answer to the open problem stated in [8] by supplying hypothesis $H(4)$ to the nonlinearity $\mathcal{F}$. In the following section, we will show that all the extra assumptions are natural in some special problems, such as the strongly damped wave equation.
3 Applications in strongly damped equations

In this section, we will apply Thm. (2.5) to the strongly damped wave equation

\[
\begin{aligned}
  u_{tt} + \eta(\Delta)^{\theta}u_t + (-\Delta)u &= f(u), \quad t > 0, \quad x \in \Omega, \\
  u(0, x) &= u_0(x), \quad u_t(0, x) = v_0(x), \quad x \in \Omega, \\
  u(t, x) &= 0, \quad t \geq 0, \quad x \in \partial\Omega,
\end{aligned}
\]  

(3.1)

Here, \( \Omega \subseteq \mathbb{R}^N (N \geq 3) \) is a bounded domain with \( C^2 \) boundary, and \( \eta > 0 \) is the strong damping coefficient. \( A = -\Delta \) denotes the negative Laplacian, which is defined in \( X = L^2(\Omega) \) as a sectorial operator with compact resolvent. \( \theta \in [\frac{1}{2}, 1] \) is a power indicator for which \( A^\theta \) and \( X^\theta (= \mathcal{D}(A^\theta)) \) denote the fractional power and fractional power space (endowed with the graph norm) of \( A \) respectively. Recall that, in this setting, \( X^0 = X, X^{1/2} = H_0^1(\Omega), \) and \( X^1 = \mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega). \)

In [5], the authors proved that \( A_\theta \) is a sectorial operator defined in \( Y \) and then \(-A_\theta\) generates an analytic semigroup \( e^{-tA_\theta} \) for each \( \theta \in [\frac{1}{2}, 1] \). Furthermore, if \( \theta \neq 1 \), then \( A_\theta \) has a compact resolvent, and correspondingly, \( e^{-tA_\theta} \) is a compact semigroup.

There are some imbedding characterization of the interpolation and extrapolation spaces of \( Y \) among which, \( Y_{\theta,-1} \) denotes the completion of \( (Y; \|A_\theta^{-1} \cdot \|) \) (see [5, 6]).

\[
Y_{\theta}^\alpha = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in X^{\frac{1}{2}+\alpha(1-\theta)} \times X^{\alpha(1-\theta)} : A^{1-\alpha}u + \eta v \in X^{\alpha} \right\}, \quad \alpha \in (\frac{1}{2}, 1],
\]  

(3.5)

\[
Y_{\theta}^\alpha = X^{\frac{1}{2}+\alpha(1-\theta)} \times X^{\alpha(1-\theta)}, \quad \alpha \in [0, \frac{1}{2}],
\]  

(3.6)

\[
Y_{\theta,-1} = X^{\frac{1}{2}-(1-\gamma)(1-\theta)} \times X^{-\frac{1}{2}+\gamma(1-\theta)}, \quad \gamma \in [0, \frac{1}{2}].
\]  

(3.7)

In order to investigate the local existence and longtime actions of the \( \varepsilon \)-regular solutions for problem (2.2), we must lay some hypotheses on the nonlinearity \( f \), i.e.

\( \text{H}_f(1): \) \( f : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function satisfying

\[
|f(u) - f(u')| \leq C|u - u'|(|u|^\rho - 1 + |u'|^{\rho - 1}), \quad \forall \, u, u' \in \mathbb{R}
\]  

(3.8)

for some \( C > 0 \) and \( 1 < \rho \leq \frac{n+2}{n-2} \).

\( \text{H}_f(2): \) \( f \) satisfies the dissipative condition

\[
\limsup_{|s| \to \infty} \frac{f(s)}{s} \leq 0.
\]  

(3.9)
Proposition 3.1 If condition (3.8) holds, then there are \( \varepsilon \in (0, \frac{1}{2p}) \), and \( \gamma \in [\rho \varepsilon, \frac{1}{2}] \), s.t. the nonlinear operator \( F : Y_{\theta-1}^{1+\varepsilon} \rightarrow Y_{\theta-1}^{\gamma} \) is \( \varepsilon \)-regular. Consequently, for any initial value \( u_0 \in Y \), there is a unique \( \varepsilon \)-regular solution \( u(\cdot) \) of (3.2) defined on its maximal interval of existence \([0, \tau)\). Moreover, if \( \gamma > \rho \varepsilon \) and \( \sup_{t \in [0, \tau)} \| u(t) \|_Y < +\infty \), then \( \tau = +\infty \).

For the proof of the above proposition and discussions about the higher regularity and other properties of the \( \varepsilon \)-regular solutions, please refer to [5, 6].

A careful investigation shows that, under conditions (3.8) and (3.9), there is an estimate for the energy of the solution, that is

\[
\left\| \left[ \begin{array}{c} \partial t u \\ \partial t v \end{array} \right] \right\|_{ Y^{\gamma} }^2 = 2\eta^t_\theta + \| A u \|_Y^2 \leq C \left( 1 + \left\| \left[ \begin{array}{c} u_0 \\ v_0 \end{array} \right] \right\|_{ Y^{\gamma} }^{\rho + 1} \right),
\]

(3.10)

where \( C > 0 \) is a constant independent of \( [u] \). Thus, the \( Y \)-norm of \( [u] \) keeps bounded on the maximal interval of existence. Furthermore, in the case \( \theta \in (\frac{1}{2}, 1) \), \( \rho \in (1, \frac{N+2}{2}) \) and \( \gamma = 1 \), \( \rho \) \( \in \) (1, \( \frac{N+2}{2} \)), we can take \( \gamma > \rho \varepsilon \) in (2.1) for \( F \), and then any \( \varepsilon \)-regular solution of (3.2) exists forever. These results can be found in [8, 9] and [10] (where \( N = 3 \) and \( \theta = 1 \)). However if the nonlinear operator \( F \) is critical, i.e. \( \rho = \frac{N+2}{\gamma} \) and \( \theta \in (\frac{1}{2}, 1) \), then according to the imbedding results obtained in the literature, \( \gamma \) can not be taken larger than \( \rho \varepsilon \). In this case, the universal existence of the \( \varepsilon \)-regular solution remains unknown. In [8], condition (3.8) has been strengthened to be \( f \in C^1(\mathbb{R}) \) satisfying

\[
\lim_{|s| \to \infty} \frac{|f(s)|}{|s|^{\frac{1}{\gamma} - 2}} \leq 0.
\]

(3.11)

By introducing notation of almost \( \varepsilon \)-regular map, the authors in [8] found that under condition (3.11), boundedness of the \( Y \)-norm leads to the global existence of the strong solution starting in \( Y \) in the case \( \theta = \frac{1}{2} \).

In the following paragraph, we will use the results established in Section 2 to show that, in the critical cases, each piecewise \( \varepsilon \)-regular solution exists globally under hypotheses (3.8) and (3.9). For this purpose, we must check all the conditions appearing in Thm. 2.5. Obviously, \( Y \) is reflexive, this is \( H(1) \). And thanks to (3.10), \( H(2) \) is also satisfied. Now, we need only to check condition \( H(4) \).

Proposition 3.2 Under hypotheses (3.8) and (3.9), the nonlinear operator \( F : Y \rightarrow Y_{(\theta)-1} \) induced by \( f \) is \( \mathcal{w} \)-continuous and bounded.

Proof. Firstly, we have

\[
\| F(u) \|_{L^{\frac{2N}{N-2}}(\Omega)} \leq C \left( \int_\Omega (1 + |u(x)|^{2^{*}(\frac{2N}{N-2})})^{\frac{N}{2N-2}} dx \right)^{\frac{1}{2}}
\]

\[
\leq C(1 + \| u \|_{L^{\frac{2N}{N-2}}(\Omega)}) \leq C(1 + \| u \|_{L^{\frac{2N}{N-2}}(\Omega)}),
\]

(3.12)

which, combining with (3.7) with \( \gamma = 0 \), yield

\[
\| F \left( \left[ \begin{array}{c} u \\ v \end{array} \right] \right) \|_{Y_{(\theta)-1}} \leq C(1 + \left\| \left[ \begin{array}{c} u \\ v \end{array} \right] \right\|_{ Y^{\frac{1}{\gamma}} }).
\]

(3.13)

This show the boundedness of \( F \).

Then we verify the \( \mathcal{w} \)-continuity of \( F \). Since \( \frac{2N}{N-2} + \frac{4}{N} < 1 \), we can select \( 0 < \varepsilon < \frac{2N}{N-2} \) small enough so that \( \frac{2N}{N-2} + \varepsilon + \frac{4}{N} < 1 \). Take \( p = \left( \frac{2N}{N-2} + \varepsilon \right)^{-1} \) and \( q = \left( \frac{2N}{N-2} - \varepsilon \right)^{-1} \), then \( 1 < p < \frac{2N}{N-2} < q < +\infty \), and \( \frac{1}{p} + \frac{4}{N} + \frac{1}{q} = 1 \). Suppose \( z_n = \left[ \begin{array}{c} u_n \\ v_n \end{array} \right] \rightarrow \left[ \begin{array}{c} u \\ v \end{array} \right] = z \) in \( Y^0 \), then \( u_n \rightarrow u \) in \( X^2 \), and consequently \( u_n \rightarrow u \) in \( L^p(\Omega) \) since \( p < \frac{2N}{N-2} \). On the other hand, by virtue of (3.8), we have

\[
\| F(u_n) - F(u) \|_{L^1(\Omega)} \leq C \int_\Omega |u_n(x) - u(x)|(1 + |u_n(x)|^{2^{*}(\frac{2N}{N-2})} + |u(x)|^{2^{*}(\frac{2N}{N-2})}) dx
\]

\[
\leq C|\Omega|^{\frac{1}{2}} \| u_n - u \|_{L^p(\Omega)}(1 + \| u_n \|_{L^{2^{*}(\frac{2N}{N-2})}(\Omega)} + \| u \|_{L^{2^{*}(\frac{2N}{N-2})}(\Omega)})
\]

(3.14)

\[
\leq C|\Omega|^{\frac{1}{2}} \| u_n - u \|_{L^p(\Omega)}(1 + \| u_n \|_{X^{1/2}} + \| u \|_{X^{1/2}}),
\]

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Thus $F(u_n) \rightarrow F(u)$ in $L^1(\Omega)$ as $n \rightarrow +\infty$. Furthermore, from the boundedness of $\{u_n\}$ in $X^\frac{1}{2}$, we can obtain the boundedness, and hence the weak precompactness of $\{F(u_n)\}$ in $L^\frac{2N}{N+2}(\Omega)$. So we can use the imbedding $L^\frac{2N}{N+2}(\Omega) \hookrightarrow L^1(\Omega)$, to deduce that the weak cluster of $\{F(u_n)\}$ in $L^\frac{2N}{N+2}(\Omega)$ is comprised of the unique strong limit $F(u)$ in $L^1(\Omega)$, consequently $F(u_n) \rightarrow F(u) \in L^\frac{2N}{N+2}(\Omega)$, and hence $X^{-\frac{1}{2}}$. On account of the construction of $F$, we finally have $F(z_n) \rightarrow F(z)$ in $X^{\theta-\frac{1}{2}} \times X^{-\frac{1}{2}}$, and hence $Y_{\theta,-1}$, this completes the proof. 

Remark 3.3 By reviewing the above proof, we can easily deduce the continuity of $F$ from $Y$ to $Y_{(\theta)-1}$, but it is useless in this paper.

Now using Thm. 2.5, we can conclude that

Theorem 3.4 If hypotheses (3.8) and (3.9) hold, then every piecewise $\varepsilon$–regular solution of problem (3.2) survive globally as the mild solution of (3.2).

Remark 3.5 In this section, we prove the global existence of the $Y$–regular solution to the strongly damped wave equation (3.1) without any other assumptions added. Therefore Thm. 3.4 can be viewed as a useful supplement to the corresponding results obtained in [8, 9].

References


