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A Network of Food Chain in Which Infectious Diseases Spread in the Middle Layer

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Abstract: In this article, we consider a three-tier nutritional food web affected by disease, and analyze its various balance points and stability. Finally, the model is analyzed in depth with specific data, together with its stability of the calculated balance point, furthermore we discuss more about the stability of the calculated balance point.

Keywords: Disease transmission; Epidemics; Food wed; Lyapunov stability

1 Introduction

In fact, the population cannot exist alone, which must interact with the related one and depend on each other. The first analysis of ecosystem stability from the perspective of community was proposed by May (1973)[1], and then perfected by Pimm (1982) et al[2]. Among them, Lotka-Volterra model[3] is the most classical and important dynamics model of the interaction in population dynamics. Studies on this model have never stopped. The earliest studies can be traced back to Wennekers, T. and Giersch, C. (1991)[4], who studied the sensitivity of the food chain of simple models. And subsequently considering inheritance processes in three food chain models (Dipak Kesh, A.K., Sarkar, A.B. Roy, 1997)[5] et al. In recent years, with the development of model theory, more and more studies have focused on specific ecosystems. For example, The external environment had an impact on the nutrient efficiency of the food chain (Peace, A., 2015)[6], Francisco Novoa-Muoz et al.[7] proposed other applications of this model variant to predict the population dynamics of the symbiotic ones.

Since ecosystem is a typical complex system, a better understanding of ecosystem requires not only long-term ecological methods, but also theories and methods of complex network science (Dobson et al., 2009)[8]. Based on complex network theory, in nearly a decade of research, the researchers mainly studied mathematical model of algae flowering (Jianyu Yao, 2011)[9], coral reef nutrition network (Briand, M., 2018)[10], microbial food web (Lingling Yan, 2018)[11], and Aquatic ecosystem (Yan Xu, 2020)[12]. In the above study, only the behavior of healthy populations in the ecosystem and the stability of the network structure were considered.

The impact of disease on the ecosystem cannot be ignored, Venturino(1993)[13] studied the effects of diseases on the structure of the food web. In many subsequent studies, a large proportion of them considered the effects of diseases on prey (Chattopadhyay et al., 1999[14]; Ebert et al., 2000[15]; Xiao and Chen, 2001[17]; Chattopadhyay et al., 2003[16]; Hethcote et al., 2004[19]; McChich et al., 2007[20]; Packer et al., 2010[18]), only a few studies have limited the impact of the disease to predators (Venturino, 2002[21]; Auger et al., 2009[22]). It is worth noting that a recent paper (Rossi et al., 2015)[23] delved the scope of disease into the prey at the bottom of the food chain, constructed differential equation model and corresponding disease-free model. Besides, consider the stability of the network and the relationship between the equilibrium solutions.

In this paper, we consider infectious diseases in a food web consisting of three trophic levels, which only affects predators in the middle of the food chain. Based on this hypothesis, the differential equation model is established to discuss the stability of the network and the influence of diseases on the population at different levels. Finally, cotton-bollworm-bird is taken as an example to simulate the actual ecosystem.

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2 The Model

In this article, we are discussing a three-level food web in the ecosystem, where W, V, X are used to express top predator, intermediate population and the bottom prey respectively. Moreover, we assume that the disease does not cross the mesosphere, which has a contact rate of ρ. Therefore, the intermediate population V is divided into two subgroups of susceptible S and infected I. On this basis, we discuss the predation relationship among the three trophic levels. And the structural relationship is shown in Fig.1.

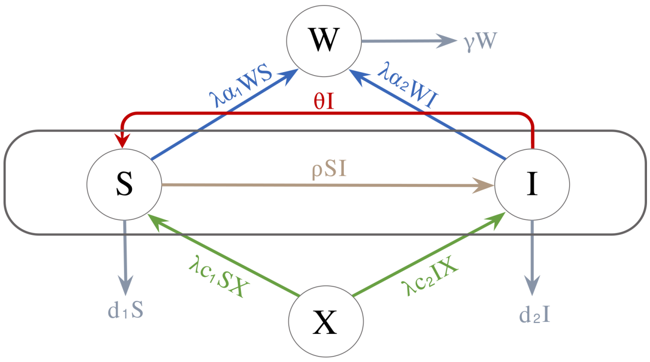

Figure 1: Model graphic of ecosystem

Overall, there are predator-bait relationships among the three trophic levels. The bottom population propagates at a net reproductive rate of m, and K stands for the environmental capacity. Due to the different success rates of prey on bait in different states, α_1 and α_2 are used to represent the predation rates of the highest trophic level for susceptible and infected middle trophic levels respectively. It makes sense that α_2 is less than α_1, because the organisms that are infected don't have the mobility and other behaviors that normal organisms do. The same principle applies between the middle trophic level and the first trophic level, but the difference is that they have different predation degrees on the bottom food, which leads to the predation rate of the middle susceptible organisms on the bottom food is higher than that of the infected organisms, namely $c_1 > c_2$. Furthermore, in the middle layer, this paper considers that there is an infectious disease in the vegetative level with a rate of ρ, and the infected has a certain recovery ability with a rate of θ. In addition, considering the actual ecosystem, it is very necessary to take the death of some organisms in the trophic level into account, which mainly occurs due to natural causes and disease. For the top trophic level organism, its natural mortality rate is γ, while for the middle trophic level, d_1 and d_2 are respectively used to represent the natural mortality rate and the disease one. Whats more, according to the law of gradual decline of ecosystem energy, we assume that the proportion of nutrient level obtaining the next nutrient level energy is λ, which makes the model established by us more consistent with the actual ecosystem. Therefore, the model we built is given by the following set of equations:

$$
\begin{aligned}
\frac{dW}{dt} &= -\gamma W + \lambda\alpha_1 WS + \lambda\alpha_2 WI \\
\frac{dS}{dt} &= \lambda c_1 XS + \theta I - \rho SI - \alpha_1 WS - d_1 S \\
\frac{dI}{dt} &= \rho SI + \lambda c_2 XI - \alpha_2 WI - \theta I - d_2 I \\
\frac{dX}{dt} &= mX(1 - \frac{X}{K}) - c_1 XS - c_2 XI
\end{aligned} \tag{1}
$$

The corresponding Jacobian matrix of (1) is:

$$
J = \begin{bmatrix} -\gamma + \lambda\alpha_1 S + \lambda\alpha_2 I & \lambda\alpha_1 W & \lambda\alpha_2 W & 0 \\ -\alpha_1 S & \lambda c_1 X - \rho I - \alpha_1 W - d_1 & \theta - \rho S & \lambda c_1 S \\ -\alpha_2 I & \rho I & \rho S + \lambda c_2 X - \alpha_2 W - \theta - d_2 & \lambda c_2 I \\ 0 & -c_1 X & -c_2 I & J_{44} \end{bmatrix} \tag{2}
$$

with

$$
J_{44} = m(1 - \frac{2X}{K}) - c_1 S - c_2 I
$$

2.1 Non-infectious disease model

At this time, we discuss the situation of no infectious disease transmission, then the two equations of the infected and susceptible in the original model are merged, so the equation of the middle trophic level is obtained as follows:

$$V = S + I, \frac{dV}{dt} = \lambda cXV - \alpha WV - dV$$

The entire food chain model then becomes:

$$\begin{aligned} \frac{dW}{dt} &= \lambda\alpha WV - \gamma W \\ \frac{dV}{dt} &= \lambda cXV - \alpha WV - dV \\ \frac{dX}{dt} &= mX(1 - \frac{X}{K}) - cVX \end{aligned} \tag{3}$$

It can be concluded that the Jacobian matrix corresponding to the disease-free model is $\hat{J}$:

$$\hat{J} = \begin{bmatrix} -\gamma + \lambda\alpha V & \lambda\alpha W & 0 \\ -\alpha V & \lambda cX - \alpha W - d_1 & \lambda cV \\ 0 & -cX & m(1 - \frac{2X}{K}) - cV \end{bmatrix} \tag{4}$$

2.1.1 Balance Point

In the $W - V - X$ phase space, this disease-free model has four equilibria:The origin $Q_0 = (0, 0, 0)$; Only the predator's equilibrium point at the bottom $Q_1 = (0, 0, K)$; The equilibrium point with no predator on the top $Q_2 = (0, V_2, X_2)$ with

$$V_2 = \frac{m}{c}(1 - \frac{d}{\lambda cK}), X_2 = \frac{d}{\lambda c}$$

Balance point of coexistence of three trophic levels $Q_3 = (W_3, V_3, X_3)$ with

$$W_3 = \frac{1}{\alpha}[\lambda cK(1 - \frac{c\gamma}{\lambda\alpha m}) - d], V_3 = \frac{\gamma}{\lambda\alpha}, X_3 = K(1 - \frac{c\gamma}{\lambda\alpha m})$$

2.1.2 Stability Analysis

The origin $Q_0 = (0, 0, 0)$ is unconditionally unstable. Now we analyze the feasibility and stability of the remaining three balance points $Q_1 = (0, 0, K), Q_2 = (0, V_2, X_2), Q_3 = (W_3, V_3, X_3)$.

Now, if

$$1 - \frac{d}{\lambda cK} < 0 \Rightarrow 1 < \frac{d}{\lambda cK} \tag{5}$$

Q_1 is stable and Q_2 is not feasible because $V_2 < 0$. At this time, the eigenvalues of the Jacobian matrix are all real numbers, so there will be no Holf branch. Instead if the converse condition holds,

$$1 - \frac{d}{\lambda cK} \geq 0 \Rightarrow 1 \geq \frac{d}{\lambda cK} \tag{6}$$

Q_2 is feasible because $V_2 \geq 0$. Meantime, there is a transcritical branch between Q_1 and Q_2.

Now, if

$$\frac{1}{\alpha}[\lambda cK(1 - \frac{c\gamma}{\lambda\alpha m}) - d] < 0 \Rightarrow \frac{\gamma}{m} > (1 - \frac{d}{\lambda cK})\frac{\lambda\alpha}{c} \tag{7}$$

Q_2 is stable, and Q_3 is not feasible because $W_3 < 0$. One more time, there will be no Holf bifurcation, owing to the following two inequalities:

$$tr(\hat{J}(Q_2)) < 0, det(\hat{J}(Q_2)) < 0 \tag{8}$$

Feasibility for Q_3 is given instead by the opposite of (7):

$$\frac{1}{\alpha}[\lambda cK(1 - \frac{c\gamma}{\lambda\alpha m}) - d] \geq 0 \Rightarrow \frac{\gamma}{m} \leq (1 - \frac{d}{\lambda cK})\frac{\lambda\alpha}{c} \tag{9}$$

Here the transcritical bifurcation between Q_2 and Q_3 can arise, for:

$$-tr(\hat{J}(Q_3)) > 0, -det(\hat{J}(Q_3)) = \frac{m}{K}\lambda\alpha^2 W_3 V_3 X_3 > 0 \tag{10}$$

At this moment,

$$\frac{m}{K}X_3(c^2V_3 + \lambda\alpha^2 W_3)V_3 > \frac{m}{K}X_3\lambda\alpha^2 W_3 \tag{11}$$

In addition, the following relationship can be easily noticed from $Q_3 = (W_3, V_3, X_3)$:

$$\lambda c X_3 - d_1 = \alpha W_3 \tag{12}$$

Specific analysis of the Jacobian matrix of $Q_3 = (W_3, V_3, X_3)$ is

$$\hat{J}(Q_3) = \begin{bmatrix} 0 & \lambda\alpha W_3 & 0 \\ -\alpha V & 0 & \lambda c V_3 \\ 0 & -cX_3 & -\frac{m}{K}X_3 \end{bmatrix}$$

Through calculation, $tr(\hat{J}(Q_3)) < 0, det(\hat{J}(Q_3)) < 0$ can arise, so Q_3 is asymptotically stable.

Theorem 1 *When the equilibrium point Q_1 is locally asymptotically stable, it is also globally asymptotically stable, using*

$$L_1 = \frac{1}{\lambda^2}W + \frac{1}{\lambda}V + (X - Kln\frac{X}{K})$$

Proof: Differentiation along the trajectory can be obtained, which leads to

$$L_1' = -\frac{1}{\lambda^2}\gamma W + (cK - \frac{d}{\lambda})V - \frac{m}{K}(X-K)^2$$

When (5) is true, it is negative definite.

Theorem 2 *Whenever the equilibrium point Q_2 is locally asymptotically stable, it is also*

$$L_2 = \frac{1}{\lambda^2}W + \frac{1}{\lambda}(V - V_2 ln\frac{V}{V_2}) + (X - X_2 ln\frac{X}{X_2})$$

Proof: When we differentiation along the trajectory can be obtained, we can get

$$L_2' = -(\frac{1}{\lambda^2}\gamma - \frac{1}{\lambda^2}\alpha V_2)W - \frac{m}{K}(X - \frac{d}{\lambda c})^2$$

which is negative exactly when (7) holds.

Theorem 3 *When the equilibrium point Q_3 is locally asymptotically stable, it is also globally asymptotically stable, using*

$$L_3 = \frac{1}{\lambda^2}(W - W_3 ln\frac{W}{W_3}) + \frac{1}{\lambda}(V - V_3 ln\frac{V}{V_3}) + (X - X_3 ln\frac{X}{X_3})$$

Proof: Differentiation along the trajectory can be obtained, which leads to

$$L_3' = -\frac{m}{K}[X - K(1 - \frac{c\gamma}{\lambda\alpha m})]^2$$

In this case, the derivative can be calculated to get $L_3' < 0$. So it is negative definite.

2.2 Subsystem with only the two lowest trophic levels

To analyze the population of the mesosphere, the main targets are S and I. But it is worth noting that, once there is no bottom-layer organism X, the existence of the mesosphere S and I will not be possible, because it is a predator-bait relationship. Therefore, we only consider the subsystem after the first equation is removed, then obtain the coexistence equilibrium point by calculation.

$$\tilde{S} = \frac{1}{\rho}(\theta + d_2 - \lambda c_2\tilde{X}), \tilde{I} = \frac{1}{c_2}[m(1 - \frac{1}{K}\tilde{X}) - \frac{c_1}{\rho}(\theta + d_2 - \lambda c_2\tilde{X})]$$

Where $\tilde{X}$ satisfies equation $B_2\tilde{X}^2 + B_1\tilde{X} + B_0 = 0$, with

$$B_2 = -\frac{\lambda m}{K}$$
$$B_1 = \lambda m + \frac{\lambda m}{\rho}(c_2d_1 - c_1d_2) + \frac{md_2}{c_2K}$$
$$B_0 = -\frac{md_2}{c_2} + \frac{1}{c_2\rho}(c_1d_2 - c_2d_1)(\theta + d_2)$$

Through the related properties of Jacobian matrix, make more study on the existence and feasibility of the equilibrium point. We get the characteristic equation $\Sigma_{i=0}^{3} A_i\Lambda^i = 0$. The coefficients of the equation are as follows:

$$A_2 = -tr(J^*) = \theta IS^{-1} > 0$$
$$A_1 = \lambda c_2^2 IX - \rho I(\theta - \rho S) + \lambda c_1^2 SX$$
$$A_0 = \lambda c_1 IX\theta(c_1 + c_2)$$

Stability can be obtained according to the Routh-Hurwitz conditions, when the following inequalities are met

$$A_0 > 0, A_1 > 0, A_2 > 0, A_2A_1 > A_1$$

3 Model Analysis

3.1 Boundedness

The total number of systems is now defined as Z(t)=W+S+I+X, the following differential equation can be easily obtained.

$$\begin{aligned}\frac{dZ}{dt} =& -\gamma W + \lambda\alpha_1 WS + \lambda\alpha_2 WI + \lambda c_1 XS + \theta I - \rho SI - \alpha_1 WS - d_1 S\\ &+ \rho SI + \lambda c_2 XI - \alpha_2 WI - \theta I - d_2 I + \frac{dX}{dt} + mX(1 - \frac{X}{K}) - c_1 XS - c_2 XI\\ =& mX(1 - \frac{X}{K}) - \alpha_1(1-\lambda)WS - \alpha_2(1-\lambda)WI - c_1(1-\lambda)XS\\ &- c_2(1-\lambda)XI - \gamma W - d_1 S - d_2 I\\ <& mX(1 - \frac{X}{K}) - \gamma W - d_1 S - d_2 I\end{aligned}$$

Now take a suitable constant η to satisfy:

$$0 < \eta < min(\gamma, d_1, d_2)$$

We can obtain

$$\begin{aligned}\frac{dZ}{dt} + \eta Z &< mX(1 - \frac{X}{K}) - \gamma W - d_1 S - d_2 I + \eta W + \eta S + \eta I + \eta X\\ &= (m+\eta)X - \frac{X}{K}X^2 + (\eta - \gamma)W + (\eta - d_1)S + (\eta - d_2)I\\ &< (m+\eta)X - \frac{m}{K}X^2 \leq \frac{-K(m+\eta)^2}{-4m}\\ &= \frac{K(m+\eta)^2}{4m} = A\end{aligned}$$

According to the theory of differential inequalities, we can get the following conclusions by integrating $Z(t)$.

$$0 \leq Z(t) < \frac{A}{\eta}(1 - e^{-\eta t}) + Z(0)e^{-\eta t}$$

On the basis of the above conclusions, let $t \rightarrow$, we can finally get $Z(t) \rightarrow A\eta^{-1}$, which shows that the total environmental quantity of the system and the quantity of each population are limited by a suitable constant

$$Z(t) \leq max\{\frac{A}{\eta}, Z(0)\}$$

3.2 Critical Point

$E_1 \equiv (0,0,0,0)$ is a clearly feasible but unstable equilibrium point. The Jacobian matrix here is:

$$J(E_1) = \begin{bmatrix} -\gamma & 0 & 0 & 0 \\ 0 & -d_1 & 0 & 0 \\ 0 & 0 & -\theta - d_2 & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$$

and the characteristic value here are $-\gamma, \lambda c_1 K - d_1, \lambda c_2 K - \theta - d_2, -m$.

$E_2 \equiv (0,0,0,K)$ is the equilibrium point with stable conditions. This equilibrium is consistent with Q_1 in the classical disease-free system. The Jacobian matrix of E_2 is:

$$J(E_2) = \begin{bmatrix} -\gamma & 0 & 0 & 0 \\ 0 & \lambda c_1 K - d_1 & \theta & 0 \\ 0 & 0 & \lambda c_2 K - \theta - d_2 & 0 \\ 0 & -c_1 K & -c_2 K & -m \end{bmatrix}$$

and the Jacobian's characteristic value at E_2 are $-\gamma, \lambda c_1 K - d_1, \lambda c_2 K - \theta - d_2, -m$, so the condition that this point needs to be balanced is

$$K < min\{\frac{d_1}{\lambda c_1}, \frac{\theta + d_2}{\lambda c_2}\} \tag{13}$$

$E_3 = (\frac{1}{\alpha}[\lambda c K(1 - \frac{c\gamma}{\lambda \alpha m}) - d], \frac{\gamma}{\lambda \alpha}, 0, K(1 - \frac{c\gamma}{\lambda \alpha m}))$ is the disease-free balance point that includes all trophic levels. If satisfied

$$c_1 K(\lambda \alpha_1 m - c_1 \gamma) \geq \alpha_1 m d_1 \tag{14}$$

then E_3 is a feasible and stable balance point. This balance is consistent with Q_3 in the disease-free system. The Jacobian matrix here is

$$J(E_3) = \begin{bmatrix} 0 & \frac{\lambda^2 \alpha_1 c_1 K m - \lambda c_1^2 K \gamma - \lambda \alpha_1 m d_1}{\alpha_1 m} & J_{13} & 0 \\ -\frac{\gamma}{\lambda} & 0 & \frac{\lambda \theta \alpha_1 - \beta \gamma}{\lambda \alpha_1} & \frac{c_1 \gamma}{\alpha_1} \\ 0 & 0 & J_{33} & 0 \\ 0 & \frac{-\lambda \alpha_1 c_1 K m + c_1^2 K \gamma}{\lambda \alpha_1 m} & \frac{-\lambda \alpha_1 c_2 K m + c_1 c_2 K \gamma}{\lambda \alpha_1 m} & -\frac{\lambda \alpha_1 m + c_1 \gamma}{\alpha_1} \end{bmatrix}$$

with

$$J_{13} = \frac{\lambda \alpha_1 \alpha_2 c_1 K m - \lambda \alpha_2 c_1^2 K \gamma - \lambda \alpha_1 \alpha_2 m d_1}{\alpha_1^2 m}$$

$$J_{33} = \frac{\alpha_1 m \rho \gamma + \lambda^2 \alpha_1^2 c_2 K m - \lambda \alpha_1 c_1 c_2 K \gamma - \lambda^2 \alpha_1 \alpha_2 c_1 K m + \lambda \alpha_2 c_1^2 K \gamma + \lambda \alpha_1 \alpha_2 m d_1 - \lambda \alpha_1^2 m(\theta + d_2)}{\lambda \alpha_1^2 m}$$

It is observed that $E_3 \equiv Q_3$, E_3 is in a situation that the disease-free balance of all trophic levels is consistent with the disease-free food chain coexistence balance Q_3, so its feasibility conditions are reduced from (14) to (9). One of the eigenvalues can be easily decomposed to give stability conditions:

$$\alpha_1 m \rho \gamma + \lambda^2 \alpha_1^2 c_2 K m + \lambda \alpha_2 c_1^2 K \gamma + \lambda \alpha_1 \alpha_2 m d_1 < \lambda \alpha_1 c_1 c_2 K \gamma + \lambda^2 \alpha_1 \alpha_2 c_1 K m + \lambda \alpha_1^2 m(\theta + d_2) \tag{15}$$

The simplified 3×3 Jacobian matrix gives the cubic characteristic equation, and the Routh-Hurwitz condition is obtained:

$$-tr(J(E_3)) = \frac{1}{\lambda \alpha_1}(\lambda \alpha_1 m - c_1 \gamma) > 0, -det(J(E_3)) = \frac{\gamma}{\lambda} W_3(\lambda \alpha_1 m - c_1 \gamma) > 0 \tag{16}$$

And it is the same as (10). In fact, considering the feasibility of (14), the Routh-Hurwitz condition corresponding to the above condition is the same as (11). Therefore, the stable value of E_3 depends on the first characteristic value, that is condition (15). The presence of parameters related to the epidemic indicates that the behavior of the ecosystem is affected by the existence of the disease.

$E_4 = (0, \frac{m(\lambda c_1 K - d_1)}{\lambda c_1^2 K}, 0, \frac{d_1}{\lambda c_1})$ represents the sub-system where only the middle layer of healthy mid-level and the bottom layer flourish. E_4 existence first needs to satisfy:

$$K \geq \frac{d_1}{\lambda c_1} \tag{17}$$

Specific analysis of the Jacobian matrix of E_4 is:

$$J(E_4)=\begin{bmatrix} -\gamma+\frac{\alpha_1 m(\lambda c_1 K-d_1)}{c_1^2 K} & 0 & 0 & 0\\ -\frac{\alpha_1 m(\lambda c_1 K-d_1)}{c_1^2 K} & 0 & \theta-\frac{\rho m(\lambda c_1 K-d_1)}{\lambda c_1^2 K} & \frac{m(\lambda c_1 K-d_1)}{c_1 K}\\ 0 & 0 & \frac{\rho m(\lambda c_1 K-d_1)}{\lambda c_1^2 K}+\frac{c_2 d_1}{c_1}-\theta-d_2 & 0\\ 0 & -\frac{d_1}{\lambda} & -\frac{c_2 d_1}{\lambda c_1} & -\frac{m d_1}{\lambda c_1 K} \end{bmatrix}$$

The Jacobian characteristic value at E_4 are:

$$\begin{aligned} \lambda_1 &= \frac{-md_1+\sqrt{m^2 d_1^2-4\lambda^2 c_1^2 d_1 K^2 m+4\lambda c_1 d_1^2 K m}}{2\lambda c_1 K}\\ \lambda_2 &= \frac{-md_1-\sqrt{m^2 d_1^2-4\lambda^2 c_1^2 d_1 K^2 m+4\lambda c_1 d_1^2 K m}}{2\lambda c_1 K}\\ \lambda_3 &= \frac{-\gamma c_1^2 K+\lambda\alpha_1 c_1 K m-\alpha_1 d_1 m}{c_1^2 K}\\ \lambda_4 &= \frac{\lambda\rho m c_1 K-\rho m d_1+\lambda c_1 c_2 d_1 K-\lambda c_1^2 K\theta-\lambda c_1^2 K d_2}{\lambda c_1^2 K} \end{aligned}$$

If and only if $m^2d_1^2+4\lambda c_1 d_1^2 Km \le 4\lambda^2 c_1^2 d_1 K^2 m$, there is a pair of complex conjugate eigenvalues with negative real parts.In the opposite case ($m^2d_1^2+4\lambda c_1 d_1^2 Km > 4\lambda^2 c_1^2 d_1 K^2 m$), these eigenvalues are all real numbers. Through observation, we can get $\lambda_2<0$. When $d_1<\lambda c_1 K$, $\lambda_1<0$ is established. Considering the feasibility of condition (17), in any case, the stability of the equilibrium point is adjusted by the remaining characteristic values. Therefore E_4 is stable if and only if the following two conditions are true:

$$\begin{aligned} \lambda\alpha_1 c_1 K m &< \gamma c_1^2 K+\alpha_1 d_1 m\\ \lambda\rho m c_1 K+\lambda c_1 c_2 d_1 K &< \rho m d_1+\lambda c_1^2 K\theta+\lambda c_1^2 K d_2 \end{aligned} \tag{18}$$

Remark 1: Since the initial complement of the first two eigenvalues of E_4 will not disappear, the Holf branch will not appear.

Remark 2: $\lambda\alpha_1 c_1 Km < \gamma c_1^2 K+\alpha d_1 m$ in condition (18) contradicts condition (8), and the feasibility condition of E_3 is reduced from (14) to (8). Therefore, there is a transcritical branch between E_3 and E_4, which corresponds to the situation of Q_2 and Q_3 in the disease-free model.

Remark 3: It is observed that the stability condition (13) of E_2 may fail. If $\frac{\theta+d_2}{\lambda c_2} > \frac{d_1}{\lambda c_1}$, then (13) is the opposite condition of (17), and (17) happens to be the feasibility condition of E_4.There is therefore a transcritical branch between these two points, and between Q_1 and Q_2 which in the case of disease models corresponding branch.

We can find out two equilibria in which the top predators disappear,

$$E_{5,6}=(0,\frac{\theta+d_2-\lambda c_2\widetilde{X}}{\rho},\frac{m}{c_2}(1-\frac{\widetilde{X}}{K})-\frac{c_1}{c_2\rho}(\theta+d_2-\lambda c_2\widetilde{X}),\widetilde{X})$$

where $\widetilde{X}$ is the root of the following equation:

$$\widetilde{A}X^2+\widetilde{B}X+\widetilde{C}=0$$

with

$$\begin{aligned} \widetilde{A} &= \lambda c_2 m\rho\\ \widetilde{B} &= -\lambda m c_2 K\rho-\lambda c_2^2 d_1 K+\lambda c_1 c_2 d_2 K-m d_2\rho\\ \widetilde{C} &= m d_2 K\rho+c_2 d_1\theta K+c_2 d_1 d_2 K-c_1 d_2\theta K-c_1 d_2^2 K \end{aligned}$$

The feasibility of $E_{5,6}$ makes the following requirements for $\widetilde{X}$:

$$\frac{\theta+d_2}{\lambda c_2}<\widetilde{X}<\frac{d_1}{\lambda c_1}$$

According to the Descartes' rule of signs, since in the equation given above

$$\widetilde{A} > 0,$$

at least one of the following two expressions must be true to ensure that the equation has root(s).

$$\widetilde{B} < 0, \widetilde{C} < 0$$

which leads to the following two inequalities:

$$\begin{aligned} &\lambda c_1 c_2 d_2 K < m d_2 \rho + \lambda m c_2 K \rho + \lambda c_2^2 d_1 K \\ &m d_2 K \rho + c_2 d_1 \theta K + c_2 d_1 d_2 K < c_1 d_2 \theta K + c_1 d_2^2 K \end{aligned}$$

The specific analysis of the Jacobian matrix of $E_{5,6}$ is:

$$J(E_{5,6}) = \begin{bmatrix} -\gamma + \lambda\alpha_1 S + \lambda\alpha_2 I & 0 & 0 & 0 \\ -\alpha_1 S & \lambda c_1 X - \rho I - d_1 & \theta - \rho & \lambda c_1 S \\ -\alpha_2 I & \rho I & 0 & \lambda c_2 I \\ 0 & -c_1 X & -c_2 X & 0 \end{bmatrix}$$

It is easy to judge that one of the characteristic values is $-\gamma + \lambda\alpha_1 S + \lambda\alpha_2 I$, and if it is less than 0, this means the upper bound of the stability of the bottom prey population:

$$X < \frac{c_2 K(\lambda\aleph_1\theta + \lambda\alpha_1 d_2 - \rho\gamma) + \lambda\alpha_2 K(m\rho - c_1\theta - c_1 d_2)}{-\lambda\alpha_1 c_2^2 K - \lambda\alpha_2 m\rho + \lambda^2\alpha_2 c_1 c_2 K}$$

The expression of the equilibrium point $E^* = (W^*, S^*, I^*, X^*)$ at which the three levels of trophic levels coexist can be obtained by the following quadratic equations of four elements:

$$\begin{aligned} &-\gamma + \lambda\alpha_1 S^* + \lambda\alpha_2 I^* = 0 \\ &\lambda c_1 X^* S^* + \theta I^* - \rho S^* I^* - \alpha_1 W^* S^* - d_1 S^* = 0 \\ &\rho S^* + \lambda c_2 X^* - \alpha_2 W^* - \theta - d_2 = 0 \\ &m(1 - \frac{X^*}{K}) - c_1 S^* - c_2 I^* = 0 \end{aligned}$$

Due to the complexity of the above equation, no special solution can be obtained. So we can get a general solution $E^* = (W^*, S^*, I^*, X^*)$ with

$$\begin{aligned} I^* &= \frac{\gamma - \lambda\alpha_1 S^*}{\lambda\alpha_2} \\ X^* &= \frac{\lambda\alpha_2 c_1 S^* K + c_2\gamma K - \lambda\alpha_1 c_2 S^* K + \lambda\alpha_2 m K}{\lambda m \alpha_2} \\ W^* &= \frac{\alpha_2\rho m S^* + \lambda\alpha_2 c_1 c_2 S^* K + c_2^2\gamma K - \lambda\alpha_1 c_2^2 S^* K + \lambda\alpha_2 c_2 m K - \alpha_2 m(\theta + d_2)}{m\alpha_2^2} \end{aligned}$$

3.3 Bistability

Now, we discuss whether some balance points can exist at the same time.

By reviewing Remark 3, we can know that the two conditions (13) and (17) are opposite, and these two conditions correspond to the feasibility conditions of E_2 and E_4 respectively. Therefore, the two balance points of E_2 and E_4 cannot exist at the same time.

A similar situation exists for E_2 and E_3. According to the feasibility of E_3, $\alpha_1 m \geq c_1\gamma$ can be obtained, and there is:

$$K > K(1 - \frac{c_1\gamma}{\lambda\alpha_1 m}) \geq \frac{d_1}{\lambda c_1} \tag{19}$$

Obviously, (19) and (13) are contradictory.

Based on the analysis of the above two balance points that cannot coexist, it is natural to discuss the coexistence between E_3 and E_4. In Remark 2 we mentioned that there is a transcritical branch between E_3 and E_4, and it is feasible to obtain E_3 through (19). And the first condition of (18) is re-expressed as follows:

$$K(1 - \frac{c_1\gamma}{\lambda\alpha_1 m}) < \frac{d_1}{\lambda c_1} \tag{20}$$

which contradicts (19).

4 Simulation

In this paper, we study the ecosystem where the food chain with three trophic levels is located, so we simulate the actual system of cotton-bollworm-bird in Jiangsu Province. First of all, we selected the parameter set $\gamma = 0.05, \alpha_1 = 0, \alpha_2 = 0.3, \lambda = 0.15, d_1 = 0.01, d_2 = 0.15, \theta = 0.01, \rho = 0.04, c_1 = 0.3, c_2 = 0.2, m = 0.4, k = 1$, to get the track chart of W, S, I and X changing with time (see Figure 2).

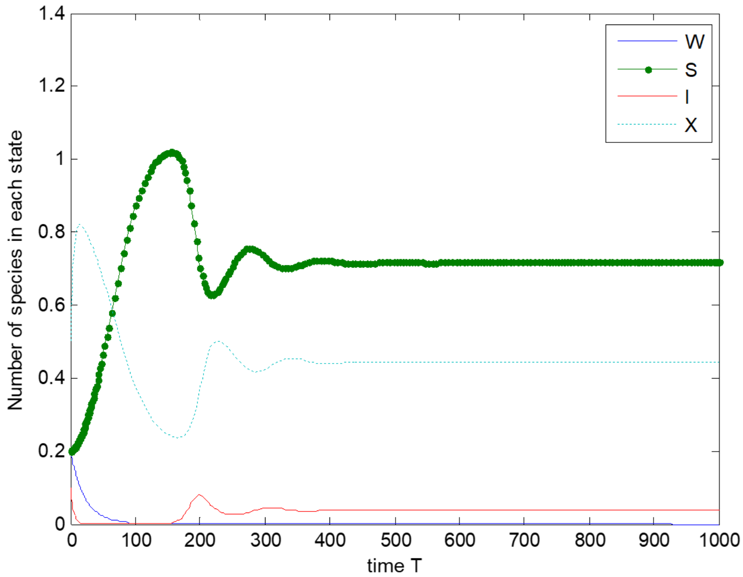

Figure 2: The trajectory of W, S, I, X over time

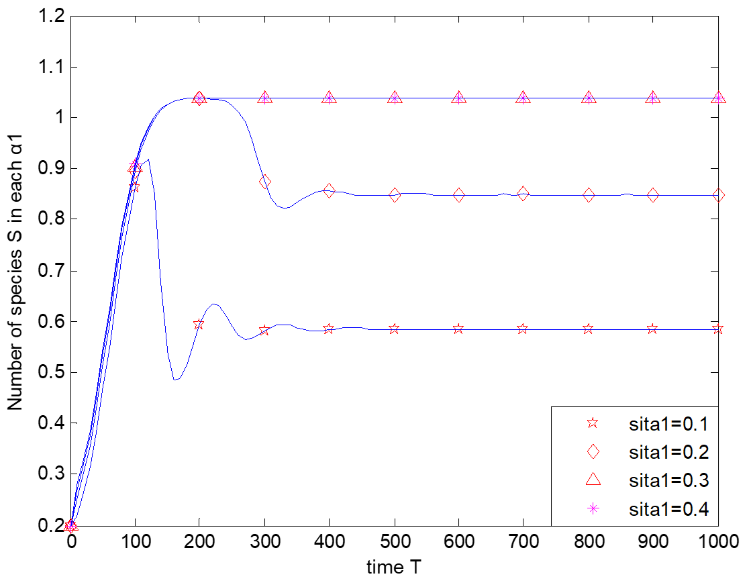

Figure 3: The trajectory of cotton bollworm S at different recovery rates over time

The following conclusions can be drawn by looking at the image. From the perspective of the trophic level of bollworm, when the ecosystem reached stability, there were still infected organisms, indicating that the disease had not disappeared and been completely cured. From the point of view of the food chain of the ecosystem, the final three nutrient levels will reach a stable state within a certain period of time, that is, the ecological balance will be reached. In terms of the process and initial state of reaching ecological balance, the number of birds gradually tends to zero, and the number of healthy bollworms gradually increases, slows down to a certain number, and then remains stable. The number of bollworms is also the largest number of organisms after the ecosystem reaches balance. This corresponds to the case of $E_{5,6}$ described earlier in this article.

In the process of increasing the recovery rate θ, the number of healthy bollworms gradually increases to 100%, while the number of sick bollworms gradually decreases until extinction. Changes in the number of various groups can be seen in Fig. 3, Fig. 4 and Fig. 5.

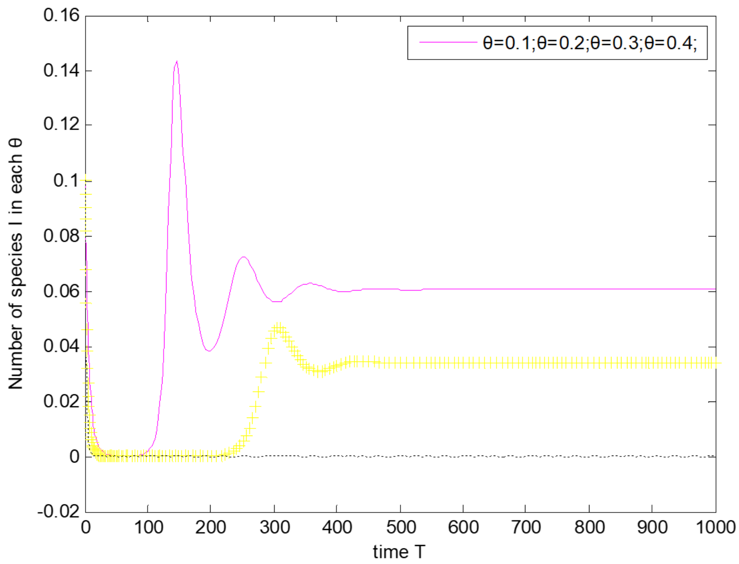

Figure 4: The trajectory of infected cotton bollworm I at different recovery rates over time

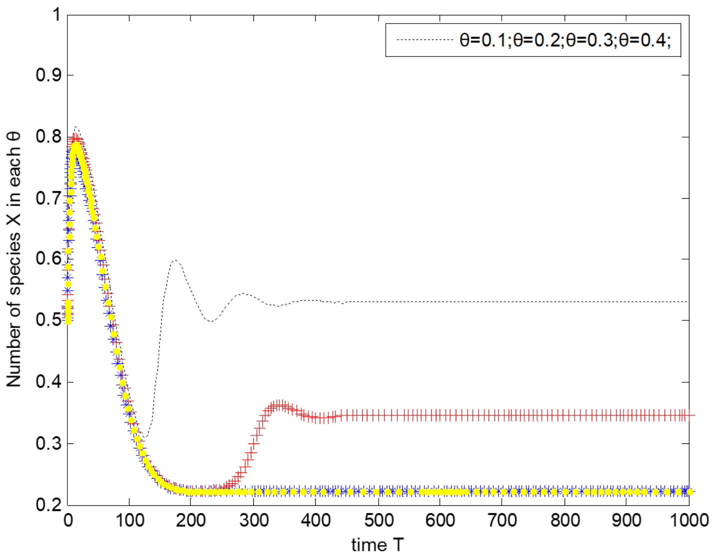

Figure 5: The trajectory of cotton yield X at different recovery rates over time

It is worth noting that when we consider changing the value of recovery rate to 0.4, the population I is dying out fairly quickly, suggesting that the epidemic prevalent in bollworm is curable under such circumstances, with a basic regeneration number less than 1. It can be seen from Fig. 5 that with the increase of θ, the quantity of bottom cotton gradually decreases, which is in line with the actual situation. The increase in the number of healthy bollworms, the natural enemies of cotton, has caused the cotton population to dwindle to a certain number, rather than disappear completely. Once cotton is extinct, the number of bollworm will inevitably decline rapidly, and the ecosystem will be faced with the imbalance.

Consider the effect of predation rate, which can be seen in Fig.6 and Fig.7.

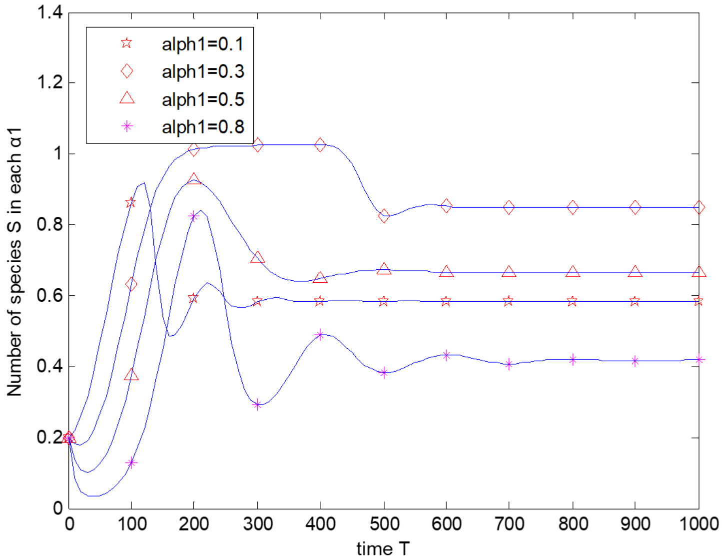

Figure 6: The trajectory of cotton bollworm S at different predation rates over time

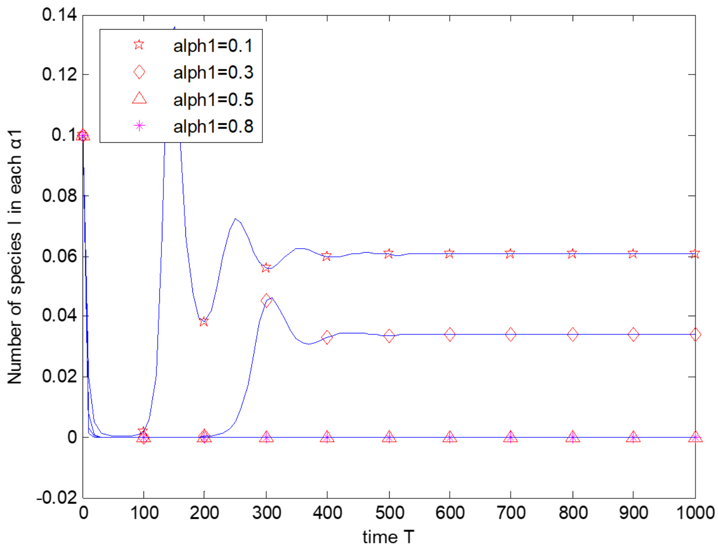

Figure 7: The trajectory of cotton bollworm I at different predation rates over time

When the predation rate $\alpha_1 < 0.5$, the population of cotton bollworm tended to be stable, and the healthy population accounted for more than half. When $\alpha_1 = 0.5$, the number of infected bollworms will no longer exist, and the number of bollworms accounts for more than 80%. When $\alpha_1 = 0.8 > 0.5$, the number of infected bollworm was still 0, while the proportion of healthy bollworm decreased to about 40%, which was caused by the intensification of intraspecie competition.

In addition, consider the impact of changes in infection rate ρ of diseases transmitted in bollworms on the number of trophic levels in the ecosystem, which can be seen in Fig.8, Fig.9 and Fig.10.

It can be found that when the infection rate gradually decreases, the number of healthy bollworms increases when

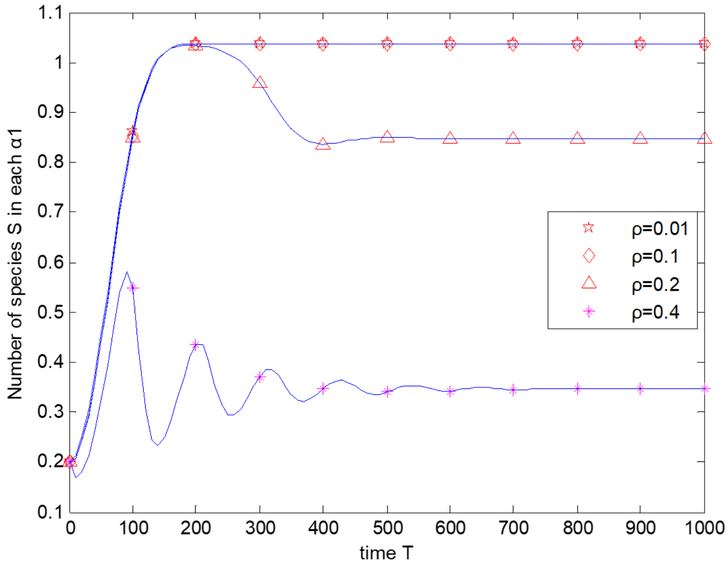

Figure 8: The trajectory of cotton bollworm S at different infection rates over time

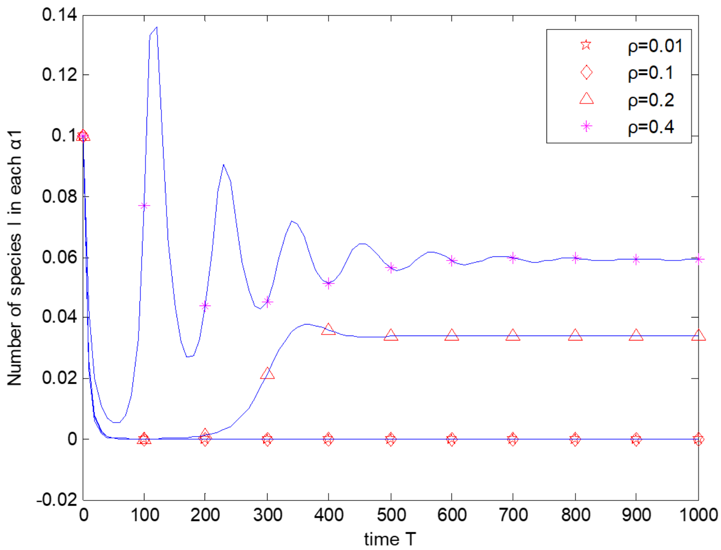

Figure 9: The trajectory of cotton bollworm I at different infection rates over time

the system reaches equilibrium, and when the infection rate drops to 0.1 or lower, the number reaches saturation, and the corresponding number of sick bollworms tends to 0 in this case. Furthermore, according to the observation of Figure 10, the quantity of the first trophic grade cotton decreases with the decrease of infection rate. When the ρ drops to 0.1 or below, the quantity of cotton decreases to a relatively low proportion, which must be greater than 0.

5 Discussion

In the general three-tier trophic food chain model, infectious diseases are introduced, which only spread in the intermediate trophic levels.

First of all, the bottom food is established using the classic Logistic model. When only the minimum nutrient level is included in the entire model, condition (5) needs to be established. At the same time, the model is stable. In other words,

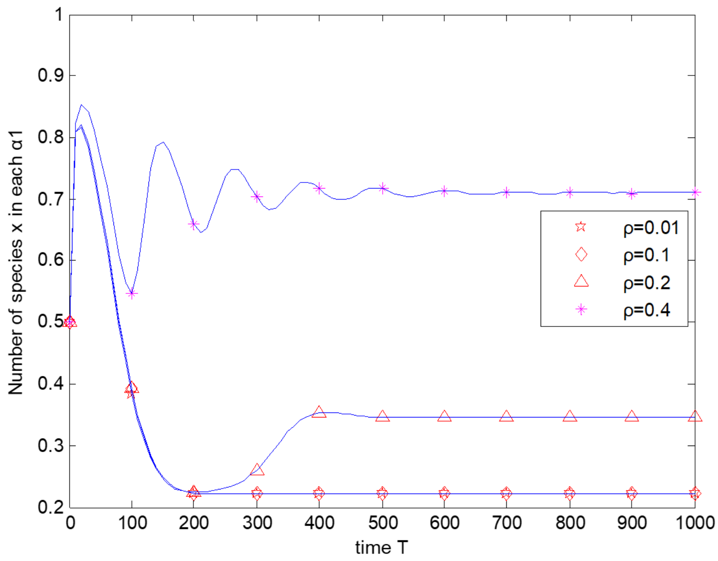

Figure 10: The trajectory of cotton yield X at different infection rates over time

in a certain limited area, when only prey is present and there are no natural predators, it can be stable for a while.

Secondly, when considering that two populations exist in the ecosystem, the condition (6) needs to be met first. In this case, we need to discuss whether there are infectious diseases or not. It turns out that all branches that exist in the classic disease-free model also appear in the system of infectious disease transmission.

Finally, look for whether there can be a bistable situation. Change the parameter set from the biological point of view, conduct in-depth analysis of the bistable situation, and then make the corresponding image.

What we have established is a food chain model, which has more complex behavior than a pure SIS infectious disease model. The main discussion is the oscillation of each equilibrium point. Comparing the two balances of E_4 and $E_{5,6}$, there are two nutrients and existence in both cases, and the latter involves the spread of infectious diseases in the middle trophic level. From the practical point of view, what we hope to get is the situation of E_4 wearing white. Take action according to the meaning of each parameter in the system, and make the actual situation as close to E_4 as possible. Now compare E_3 and $E_{5,6}$. In these two cases, E_3 contains all food chain populations and has no disease transmission, but only two trophic levels exist in $E_{5,6}$ and infectious diseases are spread among them. Whether it is from the perspective of disease transmission or from the perspective of biodiversity, we all hope to get E_3.

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