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Turing Computability of the Solution Operator of Nonlinear PP Equation

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Abstract: Turing-computability of solution operator of nonlinear pseudo-parabolic equation is studied in this paper. Firstly, we transform the equation to its integral equation by Duhamel principle. Secondly, we prove the existence and uniqueness of the solution operator of equation by principle of contraction mapping, and acclaim that its local solution is Turing-Computable in use of TTE theory. Finally, by constructing computable function, we extend the solution from the internal to the entire space. Then the solution operator of this equation is computable. The results of this paper are to study enlarge the application in computing differential equations on digital computers.

Keywords: Nonlinear pseudo-parabolic equation ; Duhamel principle ; Cauchy problem; Turing.

1 Introduction

Nonlinear evolution equations can be used to describe the mechanical, chemical recycling system, control process, biological medicine, as well as the ecological and economic system, and many other areas, its solving by computer has important applications in engineering and computer science and other fields of study, but due to its complexity, it is difficult to obtain the exact solution.So the properties of such equations, especially its boundedness, well-posedness, computability will provide a more complete theoretical basis for the application of the equation, so as to enrich the theoretical basis of computer science, and promote the development of computer software.

This article will study the following nonlinear pseudo-parabolic equation with initial boundary value problems:

$$\begin{cases} v_t - \alpha v_{xxt} - \beta v_{xx} + \gamma v_x + f(v)_x = \varphi((v_x))_x + g(v) - \alpha g(v)_{xx}, & x \in \mathbb{R}, t \in \mathbb{R}^+, \\ v(x,0) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\alpha, \beta, \gamma > 0$ are constants and $f(s)$, $\varphi(s)$ and $g(s)$ are given functions.

It is not hard to see, from equation (1), we can still get other known equations, as BBM, BBM-Burgers. More physical background can also be seen in the reference [7]. It has a great application in practice, thus some physicists and mathematicians have done a lot of research, such as its generalized solution and classical solution. [7] studied its cauchy problem. In order to study the computability of the solution of its initial value problems (1), first to rewrite its form.

Introduce transformation $v(x,t)$=$u(\frac{1}{\sqrt{a}}x,t)$, then (1) is

$$\begin{cases} u_t - u_{xxt} - \frac{\beta}{\alpha} u_{xx} + \frac{\gamma}{\sqrt{\alpha}} u_x + \frac{1}{\sqrt{\alpha}} f(u)_x = \frac{1}{\sqrt{\alpha}} \varphi(\frac{1}{\sqrt{\alpha}} u_x)_x + g(u) - g(u)_{xx}, & x \in \mathbb{R}, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $u_0(x)$=$v_0(\sqrt{\alpha}x)$.

The article is organized in the following way. In part 2, we review some definitions and lemmas. In part 3, we prove the solution operator of nonlinear pseudo-parabolic equation is computable.

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2 Preliminaries

In this section, we introduce some basic definitions and lemmas, which has something to do with the proof of the part 3.

Lemma 1 [2] 1)*In Schwarz space* $S(\mathbb{R})$*, function* $(a, \varphi) \mapsto a\varphi$ *is* $(\rho, \delta_s, \delta_s)$*-computable;* $(\varphi, t) \mapsto \varphi(t)$ *is* (δ_s, ρ, ρ)*-computable;* $(\varphi, \phi) \mapsto \varphi+\phi$ *is* $(\delta_s, \delta_s, \delta_s)$*-computable.* 2)*Function* $(\varphi, t) \mapsto V(t)\varphi$ *is* $(\delta_s, \rho, \delta_s)$*-computable.* 3)*Convolution transformation and its inverse transformation are computable.*

Lemma 2 [2]*(type conversion) Let* $\delta_i :\subseteq \sum^{\omega} \to x_i$ *be a representation of the set* x_i $(0 \le i \le k)$*. Let* $L(x_1, \cdots, x_{k-1})(x_k) = f(x_1, \cdots, x_k)$*, then* f *is* $(\delta_1, \cdots, \delta_k, \delta_0)$*-computable if and only if* L *is* $(\delta_1, \cdots, \delta_{k-1}, [\delta_k \to \delta_0])$*-computable.*

Lemma 3 [2] *The integral function:*

$$H : C(\mathbb{R}; S(\mathbb{R})) \times \mathbb{R} \times \mathbb{R} \to S(\mathbb{R}), \quad H(u, a, b) = \int_a^b u(t)dt$$

is $([\rho \to \delta_s], \rho, \rho, \delta_s)$*-computable.*

Lemma 4 [2]*(primitive recursive) Let* $\gamma :\subseteq Y \to M$ *and* $\gamma' :\subseteq Y \to M'$ *are two representations,* ν_N *is admissible representation of* N*. Then we have the following two propositions:*

1) *If* $f :\subseteq M \to M'$ *is* (γ, γ')*-computable, then* $f' :\subseteq N \times M' \times M \to M'$ *is* $(\nu_N, \gamma', \gamma, \gamma')$*-computable. Define a function*

$$g' :\subseteq N \times M \to M', g'(0, x) = f(x), g'(n+1, x) = f'(n, g'(n, x), x).$$

where $x \in M$*,*$n \in N$*, then* g' *is* (ν_N, γ, γ')*-computable.*

2) *Assuming that* $h :\subseteq M \to M$ *is* (γ, γ)*-computable, define a function*

$$H :\subseteq N \times M \to M, \quad H(0, x) = x, \quad H(n+1, x) = h \circ H(n, x) = h^{n+1}(x).$$

Then, the function H *is* (ν_N, γ, γ)*-computable.*

Definition 1 [7] *For* $T > 0$*, structure metric space* $X(T) = \{u | u \in C([0, T], H^s(\mathbb{R}))\}$ *with the norm*

$$\|u\|_{x(T)} = \max_{0 \le t \le T} \|u(\cdot, t))\|_{H^s}$$

By the Sobolev embedding theorem, there is $u, u_x \in C([0, T], L^{\infty})$ and $\|u\|_{\infty} \le K_2 \|u\|_{H^s}$, $\|u_x\|_{\infty} \le K_2 \|u\|_{H^s}$.

Lemma 5 [7] *For* $s \ge 0$*, suppose* $h(u) \in C^k(\mathbb{R})$*,* $h(0) = 0$*,* $u \in H^s \cap L^{\infty}$*, and* $k = [s] + 1$*,* $\|u\|_{\infty} \le M_0$*, then*

$$\|h(u)\|_{H^s} \le K(M_0) \|u\|_{H^s},$$

where $K(M_0)$ *is a constant only dependent on* M_0*.*

Lemma 6 [7] *Suppose* $s \ge 0$*,* $h(u) \in C^k(\mathbb{R})(k = [s] + 1)$*, if* u*,* $v \in H^s \cap L^{\infty}$*, and* $\|u\|_{\infty} \le M_0$*,* $\|v\|_{\infty} \le M_0$ *then*

$$\|h(u) - h(v)\|_{H^s} \le K_1(M_0) \|u - v\|_{H^s}.$$

These definitions and lemmas intruduced in this section are applied in the following portion frequently.

3 Main Results

In this section, we prove the main theorem of this paper. The solution operator of the problem (2) can be established as follow $K_R : H^s \to C(\mathbb{R}; H^s(\mathbb{R}))$, which defines a nonlinear mapping from initial value u_0 to the solution u.

Theorem 7 *For* $\forall t \in \mathbb{R}$*,when* $s \ge 2$*, the solution operator of the problem (5)* $K_R : H^s \to C^1(\mathbb{R}; H^s(\mathbb{R}))$ *is* $(\delta_{H^s}, [\rho \to \delta_{H^s}])$*-computable.*

To prove the theorem, first converting the differential equation into equivalent integral equation by Duhamel principle on the Sobolev space H^s, then using contraction mapping principle to prove existence and uniqueness of solution, finally proving the solution operator is computable by TTE theory and some properties of Sobolev spaces.

We have obtained equivalent integral form with the problem (5) :

$$\begin{aligned} u(x,t) =& u_0 - \frac{\beta}{\alpha}\int_0^t u(x,\tau)d\tau + \int_0^t g(u(x,\tau))d\tau \\ &+ \int_0^t G*[\frac{\beta}{\alpha}u - \frac{\gamma}{\sqrt{\alpha}}u_x - \frac{1}{\sqrt{\alpha}}f(u)_x + \frac{1}{\sqrt{\alpha}}\varphi(\frac{1}{\sqrt{\alpha}}u_x)_x](x,\tau)d\tau. \end{aligned}$$

Below, we use the contraction mapping principle to prove existence and uniqueness of solution, namely Lemma 7.

Lemma 8 *Suppose* $\forall t\in\mathbb{R}$, $s\geq 2$, $\forall u_0\in H^s(\mathbb{R})$, $g\in C^{[s]+1}(\mathbb{R})$, $g(0)=0$, $f\in C^{[s]}(\mathbb{R})$, $\varphi\in C^{[s]}(\mathbb{R})$, *then there exists* $T=T(\|u_0\|_{H^s})>0$ *satisfying the unique solution* $u\in C^1([0,T],H^s(R))$ *with problem (5).*

Proof. When $s\geq 2$,suppose $u_0\in H^s(\mathbb{R})$, and $\|u_0\|_{H^s}=M$. Structure metric space $X(T)=\{u|u\in C([0,T],H^s(\mathbb{R}))\}$, where $T>0$. Suppose $v\in Q(M,T)$,$u_0\in H^s(\mathbb{R})$, define an operator:

$$\begin{aligned} Sv(x,t) =& u_0(x) - \frac{\beta}{\alpha}\int_0^t v(x,\tau)d\tau + \int_0^t g(v(x,\tau))d\tau \\ &+ \int_0^t G*[\frac{\beta}{\alpha}v - \frac{\gamma}{\sqrt{\alpha}}v_x - \frac{1}{\sqrt{\alpha}}f(v)_x + \frac{1}{\sqrt{\alpha}}\varphi(\frac{1}{\sqrt{\alpha}}v_x)_x](x,\tau)d\tau \end{aligned}$$

Obviously, if $g\in C^{[s]+1}(\mathbb{R})$, $f\in C^{[s]}(\mathbb{R})$, $\varphi\in C^{[s]}(\mathbb{R})$, then $S:X(T)\to X(T)$.

Define a ball on $X(T)$

$$Q(M,T)=\{u|u\in X(T):\|u\|_{X(T)}\leq M+1\},$$

then $Q(M,T)$ is a Banach space. By [7], S is contraction mapping on $Q(M,T)$.

Based on contraction mapping principle, for $0<t\leq 1$,there exists a unique fixed point $u\in C^1([0,T],H^s(\mathbb{R}))$ on $Q(M,T)$,which is the solution of initial value problems (5). Proof has ended. ■

Then prove the computability of the solution of initial value problem (5), that is the proof of Theorem 1.

For $u\in H^s(\mathbb{R})(s\geq 2)$, define the solution operator:

$$\begin{aligned} S(u,u_0,t) =& u_0(x) - \frac{\beta}{\alpha}\int_0^t u(x,\tau)d\tau + \int_0^t g(u(x,\tau))d\tau \\ &+ \int_0^t G*[\frac{\beta}{\alpha}u - \frac{\gamma}{\sqrt{\alpha}}u_x - \frac{1}{\sqrt{\alpha}}f(u)_x + \frac{1}{\sqrt{\alpha}}\varphi(\frac{1}{\sqrt{\alpha}}u_x)_x](x,\tau)d\tau \end{aligned}$$

By Lemma 3.2 in [2], it is easy to prove that the operator is $([\rho\to\delta_s],\delta_s,\rho,\delta_s)$-computable.

Inference 1 Function

$$\bar{S}:C(\mathbb{R};S(\mathbb{R}))\times S(\mathbb{R})\to C(R;S(\mathbb{R})), \qquad \bar{S}(u,u_0)(t):=S(u,u_0,t),$$

then $\bar{S}$ is $([\rho\to\delta_s],\delta_s,[\rho\to\delta_s])$-computable.

Proof The result can be proved by Lemma 3.2 in [2] and Lemma 2.

Lemma 9 *Define the function*

$$v:S(R)\times N\to C(R;S(R)), v(u_0,0)=\bar{S}(0,u_0), v(u_0,j+1)=\bar{S}(v(u_0,j),u_0).$$

then v *is* $(\delta_s,v_N,[\rho\to\delta_s])$*-computable.*

Proof. The function is obtained by computable functions from primitive recursion, by Lemma 4, v is $(\delta_s,v_N,[\rho\to\delta_s])$-computable. ■

Proof of Theorem 1

Let $\omega(x,t) = u(x, t_0 + t)$, where $t \in [0,T]$, $t_0 \geq 0$. If $u(x,t_0) = \mu(x)$, then

$$\left\{\begin{array}{l} \omega_t - \alpha\omega_{xxt} - \beta\omega_{xx} + f(\omega)_x = \varphi(\omega_x)_x + g(\omega) - \alpha g(\omega)_{xx} = 0\ ,\ x \in \mathbb{R}, t > 0, \\ \omega(x,0) = \mu(x), \quad\ x \in \mathbb{R}. \end{array}\right. \tag{3}$$

If the initial value $\mu \in H^s(R)$ is given by a $\tilde{\delta}_{H^s}$-name, and since $S(R)$ is dense in $H^s(R)$, there exists a sequence $\mu_0, \mu_1 \cdots$ of Schwartz function satisfying $\|\mu_n - \mu\|_s \leq 2^{-n-2}$. For $\forall k \in N$, there is computable n_k, satisfying the following inequality

$$\|\mu_{n_k} - \mu\|_{H^s} \leq 2^{-n_k-2} \leq 2^{-k-2}.$$

Define sequence $\omega_n^0 := \overline{S}(0,\mu_n); \omega_n^{j+1} := \overline{S}(\omega_n^j, \mu_n)$. By Inference 1 and Lemma 8, the sequence $\{\omega_n^j\}$ can be computed from μ_n. By [7], on the basis of the contraction principle, the iteration $\overline{S}$ has the fixed point ω_n and ω_n is the solution of the initial problem

$$\left\{\begin{array}{l} \omega_{nt} - \alpha\omega_{nxxt} - \beta\omega_{nxx} + \gamma\omega_{nx} + f(\omega_n)_x = \varphi(\omega_{nx})_x + g(\omega_n) - \alpha g(\omega_n)_{xx}\ ,\ x \in \mathbb{R}, t > 0, \\ \omega_n(x,0) = \mu_n(x)\,, \quad x \in \mathbb{R}. \end{array}\right.$$

Since $\lim\limits_{j\to\infty} \omega_n^j = \omega_n$, we choose appropriate integer n_k, j_k, to establish a sequence $\{\omega_{n_k}^{j_k}\}_{k\in N}$, which satisfies

$$\left\|\omega_{n_k}^{j_k} - \omega_{n_k}\right\|_s \leq 2^{-k-1},$$

then the sequence $\{\omega_{n_k}^{j_k}\}_{k\in N}$ is computable. Since $n_k, j_k, \{\omega_n^j\}$ is computable, by Lemma 1, we obtain the $\tilde{\delta}_s$-name of $\omega_{n_k}^{j_k}(t)$ is computable. In the following, we certify $\{\omega_{n_k}^{j_k}\}_{k\in N}$ fast converges to ω uniformly. By (3)-(6) and (7), we can get

$$\begin{aligned}
&\|\omega_{n_k} - \omega\|_{X(T)} \\
&\leq \|\mu_{n_k} - \mu\|_{X(T)} + \| - \frac{\beta}{\alpha}\int_0^t [\omega_{n_k} - \omega]d\tau + \int_0^t [g(\omega_{n_k}) - g(\omega)]d\tau\|_{X(T)} \\
&\quad + \|\int_0^t G * (\frac{\beta}{\alpha}(\omega_{n_k} - \omega))d\tau\|_{X(T)} + \|\int_0^t G * (\frac{\gamma}{\sqrt{\alpha}}(\omega_{n_k x} - \omega_x))d\tau\|_{X(T)} \\
&\quad + \|\int_0^t G * (\frac{1}{\sqrt{\alpha}}[f(\omega_{n_k})_x - f(\omega)_x])d\tau\|_{X(T)} + \|\int_0^t G * (\frac{1}{\sqrt{\alpha}}[\varphi(\frac{1}{\sqrt{\alpha}}\omega_{n_k x})_x - \varphi(\frac{1}{\sqrt{\alpha}}\omega_x)_x])d\tau\|_{X(T)} \\
&\leq C_1\|(\mu_{n_k} - \mu)\|_{H^s} + \frac{\beta}{\alpha}T\|\omega_{n_k} - \omega\|_{X(T)} + K_1(K_2M + K_2)T\|\omega_{n_k} - \omega\|_{X(T)} \\
&\quad + \frac{\beta}{\alpha}T\|\omega_{n_k} - \omega\|_{X(T)} + \frac{|\gamma|}{\sqrt{\alpha}}T\|\omega_{n_k} - \omega\|_{X(T)} \\
&\quad + \frac{1}{\sqrt{\alpha}}K_1(K_2M + K_2)T\|\omega_{n_k} - \omega\|_{X(T)} + \frac{1}{\sqrt{\alpha}}K_1(\frac{1}{\sqrt{\alpha}}K_2M + \frac{1}{\sqrt{\alpha}}K_2)T\|\omega_{n_k} - \omega\|_{X(T)} \\
&\leq C_1\|(\mu_{n_k} - \mu)\|_{H^s} + [\frac{2\beta}{\alpha} + (1 + \frac{1}{\sqrt{\alpha}})K_1(K_2M + K_2) + \frac{|\gamma|}{\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}}K_1(\frac{1}{\sqrt{\alpha}}K_2M + \frac{1}{\sqrt{\alpha}}K_2)]T\|\omega_{n_k} - \omega\|_{X(T)}
\end{aligned}$$

Let

$$\kappa = [\frac{2\beta}{\alpha} + (1 + \frac{1}{\sqrt{\alpha}})K_1(K_2M + K_2) + \frac{|\gamma|}{\sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}}K_1(\frac{1}{\sqrt{\alpha}}K_2M + \frac{1}{\sqrt{\alpha}}K_2)],$$

choose appropriate T satisfying $\frac{C_1}{1-\kappa T} < 2$, then $\|\omega_{n_k} - \omega\|_{X(T)} \leq 2^{-k-1}$. Therefore,

$$\left\|\omega_{n_k}^{j_k} - \omega\right\|_{X(T)} \leq \left\|\omega_{n_k}^{j_k} - \omega_{n_k}\right\|_{X(T)} + \|\omega_{n_k} - \omega\|_{X(T)} \leq 2^{-k-1} + 2^{-k-1} \leq 2^{-k}.$$

Accordingly, we have proved $\{\omega_{n_k}^{j_k}\}_{k\in N}$ fast converges to ω uniformly and ω is computable.

Since the sequence $\{\omega_{n_k}^{j_k}\}_{k\in N}$ is computable, and if $\delta_S(q_k) = \omega_{n_k}^{j_k}(t)$, then $\tilde{\delta}_{H^s}\langle q_0, q_1, \cdots\rangle = \omega(t)$. That is to say, $\langle q_0, q_1, \cdots\rangle$ is the $\tilde{\delta}_{H^s}$ -name of $\omega(t)$. Hence the solution ω of the initial problem (6) is computable on $t \in [0,T]$. The solution operator mapping S is computable.

We define a $\left(\rho, \tilde{\delta}_{H^s}, \rho, \tilde{\delta}_{H^s}\right)$-computable map $F_+ : (t_0, \mu, t) \to u(t), t \in [t_0, t_0 + T]$, where $\omega(t_0) = \mu, \omega(t)$ is the solution of the initial problem (5) on $t \in [t_0, t_0 + T]$. Then we compute solution operator $u(nT)$. Define the function

$$H_+(\varphi, 0) = H_-(\varphi, 0) = \varphi$$

$$H_+(\varphi, n+1) = F_+(n \cdot T, H_+(\varphi, n), (n+1) \cdot T)$$

$$H_-(\varphi, n+1) = F_-(n \cdot T, H_+(\varphi, n), (n+1) \cdot T)$$

Because H_+ and H_- is obtained by primitive recursion from the computable function F_+ and F_-, then $H_+(\varphi, n) = u(n \cdot T)$ and $H_-(\varphi, n) = u(-n \cdot T)$ is computable.

Finally we illustrate $u(t)$ is computable.Let $n \cdot T \leq t \leq (n+1) \cdot T$, first compute $u(n \cdot T)$, then compute $F_+(n \cdot T, u(n \cdot T), t)$, therefore $u(t) = F_+(n \cdot T, u(n \cdot T), t)$ is computable.

Accordingly, the computability of solution of problem(5)is extended to $t > 0$, so for $\forall t > 0, s \geq 2$, the solution operator of nonlinear pseudo-parabolic equation (5) $K_R : H^s(R) \to C^1(R, H^s(R))$ is $(\delta_{H^s(R)}, [\rho \to \delta_{H^s(R)}])$-computable.

Summary and Outlook

This paper studies the computability of the solution operator of the nonlinear pseudo-parabolic equation. By Duhamel principle, we get its equivalent integral form. Then, we apply contraction mapping principle to analysis and theory of TTE to prove the operator can be calculated in a small interval. Lastly, by constructing computable function, we extend the solution from the local interval to the whole space ,then the solution operator of initial special equation is computable. The result of the article can lead us to further understanding of the nature and the accuracy of the physical equation,thus providing theoretical basis for computers to solve the equation. Also, it makes numerous contribution to rich the outcomes of computability.

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