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Existence and Uniqueness Classical Solution of a Class of Parabolic System with Homogenous Nonlinearity

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Abstract: In this article we consider about the semilinear parabolic system of the form $u_t - \Delta u = \mu_1 u^p + \beta u^r v^{r+1}$, $v_t - \Delta v = \mu_2 v^p + \beta u^{r+1} v^r$, where $\mu_1, \mu_2, \beta \in \mathbb{R}$, $p = 2r+1$. We use fixed point theorem to prove the existence and uniqueness classical solution of the system. Moreover, we obtain the smooth estimates of the solution.

Keywords: Parabolic system; Well-posedness, Classical solutions.

1 Introduction

In this paper we are interested in the semilinear parabolic system of the form

$$\begin{cases} u_t - \Delta u = \mu_1 |u|^{p-1} u + \beta |u|^{r-1} u |v|^{r+1}, & x \in \Omega, t > 0, \\ v_t - \Delta v = \mu_2 |v|^{p-1} v + \beta |u|^{r+1} |v|^{r-1} v, & x \in \Omega, t > 0, \\ u(x,0) = u_0, v(x,0) = v_0, & x \in \Omega, \\ u = 0, v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $p = 2r + 1$, $r > 0$, $\mu_1, \mu_2, \beta \in \mathbb{R}$, Ω is a domain of $\mathbb{R}^n$, $n = 1, 2, 3$.

Before we motivate and formulate the main question, it is convenient to recall some known facts from the case of scalar semilinear parabolic equations. Consider the scalar problem

$$\begin{cases} u_t - \Delta u = f(u), & x \in \Omega, t > 0, \\ u(x,0) = u_0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{2}$$

where f is a C^1 function and Ω is an arbitrary domain of $\mathbb{R}^n$. It is well known that problem (2) possesses a local in time solution if $u_0 \in L^\infty(\Omega)$. If $u_0 \notin L^\infty(\Omega)$, it is known that the existence of solutions heavily depends on the balance between the growth rate of f and the singularity of u_0. [15] studied (2) when $f(u) = |u|^{p-1}u$ with $p > 1$ and Ω is a smoothly bounded domain of $\mathbb{R}^n$. Let us define $q_c = \frac{n(p-1)}{2}$, then their results on local solvability read as follows

(i) If $q > q_c$, then $L^q(\Omega)$ is a well-posedness space, in the sense that for all $u_0 \in L^q(\Omega)$, $1 \le q < \infty$, there exists a unique classical L^q-solution and that $u \le 0$ whenever $u_0 \le 0$. If u is any local in time solution, then $sup_{t\in[0,T)} \|u(t)\|_q = \infty$.

(ii) If $1 \le q < q_c$ and $p > 1 + \frac{2}{n}$, then $L^q(\Omega)$ is not a well-posedness space. More precisely, there exists a nonnegative initial data $u_0 \in L^q(\Omega)$, such that no nonnegative classical L^q-solution on $[0, T)$ for any $T > 0$. For example $u_0(y) = |y|^{-\alpha} \chi_{B(0,\rho)}(y)$, $\alpha \in (0, \frac{n}{q})$, $\rho > 0$. And for some u_0, there exist a local in time u, such that $sup_{t\in[0,T)} \|u(t)\|_q < \infty$.

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Now consider the following more general reaction-diffusion system

$$\begin{cases} u_t - \Delta u = f(u,v), & x \in \Omega, t > 0, \\ v_t - \Delta v = g(u,v), & x \in \Omega, t > 0, \\ u(x,0) = u_0, v(x,0) = v_0, & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{3}$$

The paper [13] studied (3), where $f(u,v) = |v|^{p_1-1}v$, $g(u,v) = |u|^{p_2-1}u$, $p_1, p_2 \geq 1$, Ω is a smoothly bounded domain of $\mathbb{R}^n$, $u_0, v_0 \in L^q$ for any $1 \leq q < \infty$. This is called a weakly coupled system. They proved that when $p_1, p_2 \geq 1$, $q \geq 1$

(i) If $q > 1$ and $\max(p_1,p_2) \leq 1 + \frac{2q}{n}$, then there exists a unique maximal solution (u,v). Moreover, $u, v \geq 0$ if $u_0, v_0 \geq 0$.

(ii) If $max(p_1,p_2) > 1 + \frac{2q}{n}$, then there exists an initial data $(u_0,v_0) \in L^q(\Omega) \times L^q(\Omega)$, $u_0, v_0 \geq 0$, such that there is no nonnegative solution (u,v) of (3).

In particular, for a system of two reaction-diffusion equations coupled by power nonlinearities. The paper [4] considered the boundedness and blow up of the semilinear parabolic system

$$\begin{cases} u_t = \Delta u + v^p, & x \in \mathbb{R}^n, t > 0, \\ v_t = \Delta v + u^p, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $n \geq 1$, $p > 0$, $q > 0$, $u_0(x)$, $v_0(x)$ are nonnegative, continuous and bounded real functions. They proved that when p, q are different from zero and $p > 1$ or $q < 1$

(i) If $0 < pq < 1$ and $(u_0,v_0) \neq (0,0)$, then (4) has a unique solution.

(ii) If $0 < pq < 1$ and $(u_0,v_0) = (0,0)$, the set of nontrivial nonnegative solutions is given by

$$u(t;s) = c_1(t-s)_+^{\alpha},\ v(t;s) = c_2(t-s)_+^{\beta},$$

where $(r)_+ = \max\{r, 0\}$, s is any nonnegative real constant and $\alpha = \frac{p+1}{1-pq}$, $\beta = \frac{q+1}{1-pq}$, $c_1 = (1-pq)^{\frac{p+1}{1-pq}}(p+1)^{-\frac{1}{1-pq}}(q+1)^{-\frac{p}{1-pq}}$, $c_2 = c_1^q/\beta$.

(iii) If $pq \geq 1$, there is a unique solution of (4).

In [8], the authors have studied about the global existence of the Cauchy problem for a two space dimensional parabolic equation with square exponential nonlinearity with the same method.

For the statement of main results, let us introduce the Sobolev exponent:

$$p_s := \begin{cases} \infty, & n = 1,2, \\ \frac{n+2}{n-2}, & n \geq 3, \end{cases}$$

where $f(u,v)$, $g(u,v)$ are C^1 functions with a superlinear growth.

The aim of this paper is to prove whether (1) has a unique classical L^q-solution, such a result is called the well-posedness of the semilinear parabolic system (1).

Theorem 1 *Let $p > 1$, $(u_0,v_0) \in L^q(\Omega) \times L^q(\Omega)$, $1 \leq q < \infty$, $q > q_c := \frac{n(p-1)}{2}$. Then there exists $T = T(\|(u_0,v_0)\|_q) > 0$ such that problem (1) possesses a unique classical L^q-solution in $[0,T)$ and the following smoothing estimate is true:*

$$\|(u,v)\|_k \leq C\|(u_0,v_0)\|_k t^{-\alpha_k},\ \alpha_k := \frac{n}{2}(\frac{1}{q} - \frac{1}{k}), \tag{5}$$

for all $t \in (0,T)$ and $k \in [q,\infty]$ with $C = C(n,p,q) > 0$. In addition, $u \geq 0, v \geq 0$ provided $u_0 \geq 0, v_0 \geq 0$.

2 Preliminaries

In this section we introduce the definitions of classical X-solution, weak L^1_δ-solution, mild L^q- solutions and well-posedness of a class of the semilinear parabolic system (3). Moreover, we recall some properties of the fundamental solution of the semilinear parabolic equation and give some preliminary L^p-L^q-estimates.

Definition 1 *Given a Banach space X of function defined in Ω, $(u_0, v_0) \in X \times X$ and $T \in (0, \infty]$, we say that the function $(u, v) \in C([0, T), X \times X)$ is a solution (more precisely a classical X-solution) of* (3) *in $[0, T)$ if $u, v \in C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times (0, T))$, $(u(0), v(0)) = (u_0, v_0)$ and (u, v) is a classical solution of* (3) *for $t \in (0, T)$. If Ω is unbounded, then we require $(u, v) \in L^\infty_{loc}((0, T), L^\infty(\Omega) \times L^\infty(\Omega))$.*

We say that (1) is well-posed in $X \times X$, if given $(u_0, v_0) \in X \times X$, there exist $T > 0$ and a unique classical X-solution of (1) in $[0, T]$.

Remark 2 *If (u, v) is a classical solution of the system $u_t + Au(s) = f(u(s), v(s))$ and $v_t + Av(s) = g(u(s), v(s))$ in $[0, T)$, then applying the operator $e^{-(t-s)A}$ to the equation, integrating in $s \in (\tau, t)$ and using $\frac{d}{ds}(e^{-(t-s)A}u(s)) = e^{-(t-s)A}(u_t(s) + Au(s))$, $\frac{d}{ds}(e^{-(t-s)A}v(s)) = e^{-(t-s)A}(v_t(s) + Av(s))$, we obtain the variational-of-constants formula*

$$\begin{aligned} u(t) &= e^{-(t-s)A}u(\tau) + \int_\tau^t e^{-(t-s)A} f(u(s), v(s))ds \\ v(t) &= e^{-(t-s)A}v(\tau) + \int_\tau^t e^{-(t-s)A} g(u(s), v(s))ds \end{aligned} \tag{1}$$

Definition 2 *Any function $(u, v) \in C([0, T), L^q \times L^q(\Omega))$ satisfying $f(u, v), g(u, v) \in L^1_{loc}((0, T), L^1(\Omega) + L^\infty(\Omega))$, $(u(0), v(0)) = (u_0, v_0)$ and* (1) *is called a mild L^q-solution of* (3)*.(If $q = \infty$, then we modify this definition in the same way as in the case of classical solutions.)*

Definition 3 *Consider problem* (3) *with f, g nonnegative and $u_0 \geq 0$, $v_0 \geq 0$. We say that (u, v) is an integral solution of* (3) *in $[0, T)$ if $u := \Omega \times [0, T) \to [0, \infty]$, $v := \Omega \times [0, T) \to [0, \infty]$ are measurable, finite a.e. and*

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \int_\Omega G(x, y, t)u_0(y)dy + \int_0^t \int_\Omega G(x, y, t-s)f(u(y, s), v(y, s))dyds \\ \int_\Omega G(x, y, t)v_0(y)dy + \int_0^t \int_\Omega G(x, y, t-s)g(u(y, s), v(y, s))dyds \end{pmatrix} \tag{2}$$

for a.e. $(x, t) \in Q_T$, where G is the Dirichlet heat kernel.

Remark 3 *If $(u_0, v_0) \in L^q(\Omega) \times L^q(\Omega)$ is nonnegative and (u, v) is a mild L^q-solution of* (3)*, then (u, v) is also an integral solution of* (3)*.*

Definition 4 *We denote the distance to boundary function by*

$$\delta := dist(x, \partial\Omega),\ x \in \Omega.$$

Assume that Ω is bounded and $(u_0, v_0) \in L^1_\delta(\Omega) \times L^1_\delta(\Omega)$. A function $(u, v) \in C([0, T), L^1_\delta(\Omega) \times L^1_\delta(\Omega))$ is called a weak L^1_δ-solution of (3) *in $[0, T)$ if the functions (u, v), $\delta f(u, v)$, $\delta g(u, v)$ belong to $L^1_{loc}((0, T), L^1(\Omega)) \times L^1(\Omega))$, $(u(0), v(0)) = (u_0, v_0)$ and*

$$\int_\tau^t \int_\Omega (f(u, v), g(u, v))\varphi = -\int_\tau^t \int_\Omega (u, v)(\varphi_t + \Delta\varphi) - \int_\Omega (u(\tau), v(\tau))\varphi(\tau)) \tag{3}$$

for any $0 < \tau < t < T$ and any $\varphi \in C^2(\overline{\Omega} \times [\tau, t])$ such that $\varphi = 0$ on $\partial\Omega \times [\tau, t]$ and $\varphi(t) = 0$.

The uniqueness of weak solution and the equivalence between the weak solution and the mild solution of the linear problem

$$\begin{cases} u_t - \Delta u = f(x), & x \in \Omega, t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \end{cases}$$

are proved in [15, Proposition 48.9, Corollary 48.10, Corollary 48.11]. The following similar results corresponding to the linear system

$$\begin{cases} u_t - \Delta u = f(x), & x \in \Omega, t \in (0,T), \\ v_t - \Delta v = g(x), & x \in \Omega, t \in (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \\ u = (x,t) = 0, v(x,t) = 0, & x \in \partial\Omega, t \in (0,T) \end{cases} \tag{4}$$

can be proved with the same method.

Proposition 4 *Let Ω be a bounded domain of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$ and let $(u_0, v_0) \in L^1_\delta(\Omega) \times L^1_\delta(\Omega)$.*

(i) *If $f, g \in L^1_{loc}([0,T), L^1_\delta(\Omega) \times L^1_\delta(\Omega))$, then problem (4) possesses a unique weak L^1_δ-solution. Moreover $(u,v) \in L^1_{loc}([0,T), L^1(\Omega) \times L^1(\Omega))$.*

(ii) *If $f, g \in L^1_{loc}((0,T), L^1_\delta(\Omega) \times L^1_\delta(\Omega))$ and problem (4) possesses a weak L^1_δ-solution, then $f, g \in L^1_{loc}([0,T), L^1_\delta(\Omega) \times L^1_\delta(\Omega))$.*

Corollary 5 *Let Ω be a bounded domain of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$, $u_0, v_0 \in L^1_\delta(\Omega)$, $u, v \in L^1_{loc}(\Omega \times (0,T))$, and $f, g : \Omega \times \Omega \times (0,T) \to \mathbb{R}$ be measurable. Assume that $u_0, v_0, u, v, f, g \geq 0$,*

(i) *If $f, g \in L^1_{loc}((0,T), L^1_\delta(\Omega) \times L^1_\delta(\Omega))$ and (u,v) is a weak L^1_δ-solution of (4), then it is an integral solution of (4).*

(ii) *If (u,v) is an integral solution of (4), then $f, g \in L^1_{loc}((0,T), L^1_\delta(\Omega) \times L^1_\delta(\Omega))$ and (u,v) is a weak L^1_δ-solution of (4).*

Corollary 6 *Let Ω be a bounded domain of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$ and let (u,v) be a mild L^q-solution of (4), then (u,v) is a weak L^1_δ-solution of (4).*

3 Proof of Theorem 1

In this section, we prove the well-posedness of (1). In other words, we prove the uniqueness of the local in time solution of (1). At the beginning of this section we introduce some properties of the heat semigroup e^{-tA}.

Lemma 7 *Let $(e^{-tA})_{t\geq 0}$ be the heat semigroup in $\mathbb{R}^n$ and $G_t(x) = G(x,t)$ the Guassian heat kernel. We have the following properties*

(i) $\|G_t\|_1 = 1$ *for all* $t > 0$.

(ii) *If* $\Phi \geq 0$, *then* $e^{-tA}\Phi \geq 0$ *and* $\|e^{-tA}\Phi\|_1 = \|\Phi\|_1$.

(iii) *If* $1 \leq q \leq \infty$, *then* $\|e^{-tA}\Phi\|_q \leq \|\Phi\|_q$ *for all* $t > 0$.

(iv) *If* $1 \leq p < q \leq \infty$ *and* $1/r = 1/p - 1/q$, *then* $\|e^{-tA}\Phi\|_q \leq (4\pi t)^{-n/(2r)}\|\Phi\|_p$ *for all* $t > 0$.

(v) *For an arbitrary domain $\Omega \subset \mathbb{R}^n$, assertions (iii) and (iv) remain valid of e^{-tA} is replaced with the Dirichlet heat semigroup in Ω.*

The proof of this Lemma can be found in [15, Proposition 48.3].

Proof of Theorem 1. It is divided into six steps.

Step 1. Fixed-point argument. To handle the singularity of the initial data, the idea is to introduce a Banach space of functions with a temporal weight which has a suitable decay as $t \to 0$. We may assume that$\|(u_0, v_0)\|_q > 0$. Let $T > 0$ be small and consider the Banach space

$$Y_T := \left\{(u,v) \in L^\infty_{loc}\left((0,T), L^{pq}(\Omega) \times L^{pq}(\Omega)\right) : \|(u,v)\|_{Y_T} < \infty\right\},$$

the norm in the space Y_T is defined by

$$\|(u,v)\|_{Y_T} := \sup_{0<t<T} t^\alpha \|(u(t), v(t))\|_{pq},$$

where $\alpha := \frac{n(p-1)}{2pq} < \frac{1}{p} < 1$. Choose $M > \|(u_0, v_0)\|_q$ and let $B_M = B_{M,T}$ denote the closed ball in Y_T with center 0 and radius M. We will use the Banach fixed point theorem for the mapping $\Phi_{(u_0,v_0)} : B_M \to B_M$, where

$$\Phi_{(u_0,v_0)}(u,v)(t) := \begin{pmatrix} e^{-tA}u_0 + \int_0^t e^{-(t-s)A}(\mu_1|u(s)|^{p-1}u(s) + \beta|u(s)|^{r-1}u(s)|v(s)|^{r+1})ds \\ e^{-tA}v_0 + \int_0^t e^{-(t-s)A}(\mu_2|v(s)|^{p-1}v(s) + \beta|u(s)|^{r+1}|v(s)|^{r-1}v(s))ds \end{pmatrix}.$$

For the convenience of calculation, we denote

$$\Phi_{(u_0,v_0)}(u,v)(t) = \begin{pmatrix} \Phi^1_{(u_0,v_0)}(u,v)(t) \\ \Phi^2_{(u_0,v_0)}(u,v)(t) \end{pmatrix}.$$

Using the L^p-L^q-estimate and Minkovski inequality, we obtain for any $(u_1, v_1), (u_2, v_2) \in B_M$ and $(u_0^1, v_0^1), (u_0^2, v_0^2) \in L^q(\Omega) \times L^q(\Omega)$.

$$\begin{aligned}
& t^\alpha \|\Phi^1_{(u_0^1,v_0^1)}(u,v)(t) - \Phi^1_{(u_0^2,v_0^2)}(u,v)(t)\|_{pq} \\
&= t^\alpha \|e^{-tA}(u_0^1 - u_0^2) + \int_0^t e^{-(t-s)A}(\mu_1|u_1(s)|^{p-1}u_1(s) - \mu_1|u_2(s)|^{p-1}u_2(s) \\
&\quad + \beta|u_1(s)|^{r-1}u_1(s)|v_1(s)|^{r+1} - \beta|u_2(s)|^{r-1}u_2(s)|v_2(s)|^{r+1})ds\|_{pq} \\
&\leq t^\alpha \|e^{-tA}(u_0^1 - u_0^2)\|_{pq} + t^\alpha \int_0^t \|e^{-(t-s)A}(\mu_1|u_1(s)|^{p-1}u_1(s) - \mu_1|u_2(s)|^{p-1}u_2(s))\|_{pq} ds \\
&\quad + t^\alpha \int_0^t \|e^{-(t-s)A}(\beta|u_1(s)|^{r-1}u_1(s)|v_1(s)|^{r+1} - \beta|u_2(s)|^{r-1}u_2(s)|v_2(s)|^{r+1})\|_{pq} ds \\
&\leq (4\pi)^{-\alpha}\|u_0^1 - u_0^2\|_q + C(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}\||u_1(s)|^{p-1}u_1(s) - |u_2(s)|^{p-1}u_2(s)\|_q ds \\
&\quad + |\beta|(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}\||u_1(s)|^{r-1}u_1(s)|v_1(s)|^{r+1} - |u_2(s)|^{r-1}u_2(s)|v_2(s)|^{r+1}\|_q ds \\
&\leq (4\pi)^{-\alpha}\|u_0^1 - u_0^2\|_q + CC'(p)(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}(\|u_1\|_{pq}^{p-1} + \|u_2\|_{pq}^{p-1})\|u_1 - u_2\|_{pq} ds \\
&\quad + |\beta|(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}\||u_1(s)|^{r-1}u_1(s)|v_1(s)|^{r+1} - |u_2(s)|^{r-1}u_2(s)|v_2(s)|^{r+1}\|_q ds \\
&\leq (4\pi)^{-\alpha}\|u_0^1 - u_0^2\|_q + CC_1(p)M^{p-1}t^\alpha \int_0^t (t-s)^{-\alpha}s^{-(p-1)\alpha}\|u_1(s) - u_2(s)\|_{pq} ds \\
&\quad + |\beta|(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}\||u_1(s)|^{r-1}u_1(s)|v_1(s)|^{r+1} - |u_2(s)|^{r-1}u_2(s)|v_2(s)|^{r+1}\|_q ds \\
&= (4\pi)^{-\alpha}\|u_0^1 - u_0^2\|_q + CC_1(p)M^{p-1}t^\alpha \int_0^t (t-s)^{-\alpha}s^{-(p-1)\alpha}\|u_1(s) - u_2(s)\|_{pq} ds \\
&\quad + |\beta|(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}\||u_1(s)|^{r-1}u_1(s)(|v_1(s)|^{r+1} - |v_2(s)|^{r+1})\|_q ds \\
&\quad + |\beta|(4\pi)^{-\alpha}t^\alpha \int_0^t (t-s)^{-\alpha}\||v_2(s)|^{r+1}(|u_1(s)|^{r-1}u_1(s) - |u_2(s)|^{r-1}u_2(s))\|_q) ds
\end{aligned}$$

$$
\begin{aligned}
&\leq (4\pi)^{-\alpha}\|u_0^1-u_0^2\|_q + CC_1(p)M^{p-1}t^{\alpha}\int_0^t (t-s)^{-\alpha}s^{-(p-1)\alpha}\|u_1(s)-u_2(s)\|_{pq}ds\\
&\quad + |\beta|(4\pi)^{-\alpha}t^{\alpha}\int_0^t (t-s)^{-\alpha}\|u_1(s)\|_{pq}^r\||v_1(s)|^{r+1}-|v_2(s)|^{r+1}\|_{\frac{pq}{r+1}}ds\\
&\quad + |\beta|(4\pi)^{-\alpha}t^{\alpha}\int_0^t (t-s)^{-\alpha}\|v_2(s)\|_{pq}^{r+1}\||u_1(s)|^{r-1}u_1(s)-|u_2(s)|^{r-1}u_2(s)\|_{\frac{pq}{r}})ds\\
&\leq (4\pi)^{-\alpha}\|u_0^1-u_0^2\|_q + CC_1(p)M^{p-1}t^{\alpha}\int_0^t (t-s)^{-\alpha}s^{-(p-1)\alpha}\|u_1(s)-u_2(s)\|_{pq}ds\\
&\quad + |\beta|(4\pi)^{-\alpha}t^{\alpha}\int_0^t (t-s)^{-\alpha}C(r)\|u_1(s)\|_{pq}^r\|v_1(s)-v_2(s)\|_{pq}(\|v_1(s)\|_{pq}^r+\|v_2(s)\|_{pq}^r)ds\\
&\quad + |\beta|(4\pi)^{-\alpha}t^{\alpha}\int_0^t (t-s)^{-\alpha}C(r)\|v_2(s)\|_{pq}^{r+1}\|u_1(s)-u_2(s)\|_{pq}(\|u_1(s)\|_{pq}^{r-1}+\|u_2(s)\|_{pq}^{r-1})ds
\end{aligned}
\tag{5}
$$

In particular, choosing (u_0^2,v_0^2) and $(u_2,v_2)=(0,0)$ in (5), we have

$$
\begin{aligned}
&\|\Phi^1_{(u_0^1,v_0^1)}(u_1,v_1)\|_{Y_T}\\
&\leq (4\pi)^{-\alpha}\|u_0^1\|_q + CC_1(p)M^{p-1}t^{\alpha}\int_0^t (t-s)^{-\alpha}s^{-(p-1)\alpha}\|u_1(s)\|_{pq}ds\\
&\quad + |\beta|C_2t^{\alpha}\int_0^t (t-s)^{-\alpha}\|u_1(s)\|_{pq}^r\|v_1(s)\|_{pq}^{r+1}ds\\
&\leq (4\pi)^{-\alpha}\|u_0^1\|_q + \sup_{0<t<T}|\mu_1|C_1(p)M^{p-1}t^{\alpha}\|u_1\|_{Y_T}\int_0^t (t-s)^{-\alpha}s^{-p\alpha}ds\\
&\quad + \sup_{0<t<T}|\beta|C_2t^{\alpha}\|u_1\|_{Y_T}^r\|v_1\|_{Y_T}^{r+1}\int_0^t (t-s)^{-\alpha}s^{-p\alpha}ds\\
&\leq (4\pi)^{-\alpha}\|u_0^1\|_q + |\mu_1|C_1(p,\alpha)M^{p-1}T^{1-p\alpha}\|u_1\|_{Y_T} + |\beta|C_2(p,\alpha)T^{1-p\alpha}\|u_1\|_{Y_T}^r\|v_1\|_{Y_T}^{r+1}\\
&\leq (4\pi)^{-\alpha}M + |\mu_1|C_1(p,\alpha)M^pT^{1-p\alpha} + |\beta|C_2(p,\alpha)M^pT^{1-p\alpha}.
\end{aligned}
$$

Let $T_0=T_0(M,n,p,q)>0$ be such that

$$
\begin{aligned}
&|\mu_1|C_1(p,\alpha)M^{p-1}T^{1-p\alpha} < \min\{\frac{1}{3}-\frac{1}{2}(4\pi)^{-\alpha},\frac{1}{6}\},\\
&|\beta|C_2(p,\alpha)M^{p-1}T^{1-p\alpha} < \min\{\frac{1}{3}-\frac{1}{2}(4\pi)^{-\alpha},\frac{1}{6C(r)}\}.
\end{aligned}
\tag{6}
$$

The above estimate implies

$$\|\Phi^1_{(u_0^1,v_0^1)}(u_1,v_1)\|_{Y_T} < \tfrac{2}{3}M \quad \text{for any } T\leq T_0.$$

Similarly as above we obtain

$$\|\Phi^2_{(u_0^1,v_0^1)}(u_1,v_1)\|_{Y_T} < \tfrac{2}{3}M \quad \text{for any } T\leq T_0.$$

Then

$$\|\Phi_{(u_0^1,v_0^1)}(u_1,v_1)\|_{Y_T} < \frac{2\sqrt{2}}{3}M \quad \text{for any } T\leq T_0, \tag{7}$$

hence $\Phi_{(u_0,v_0)}$ maps B_M into B_M for $T\leq T_0$. Choosing $(u_0^1,v_0^1)=(u_0^2,v_0^2)=(u_0,v_0)$ in (5), we obtain

$$
\begin{aligned}
&\|\Phi^1_{(u_0,v_0)}(u_1,v_1)-\Phi^1_{(u_0,v_0)}(u_2,v_2)\|_{Y_T}\\
&\leq |\mu_1|C_1(p,\alpha)M^{p-1}T^{1-p\alpha}\|u_1-u_2\|_{Y_T}\\
&\quad + |\beta|C_2(p,\alpha)T^{1-p\alpha}C(r)\|u_1\|_{Y_T}^r\|v_1-v_2\|_{Y_T}(\|v_1\|_{Y_T}^r+\|v_2\|_{Y_T}^r)\\
&\quad + |\beta|C_2(p,\alpha)T^{1-p\alpha}C(r)\|v_2\|_{Y_T}^{r+1}\|u_1-u_2\|_{Y_T}(\|u_1\|_{Y_T}^{r-1}+\|u_2\|_{Y_T}^{r-1})\\
&\leq |\mu_1|C_1(p,\alpha)M^{p-1}T^{1-p\alpha}\|u_1-u_2\|_{Y_T} + |\beta|C_2(p,\alpha)T^{1-p\alpha}M^{p-1}C(r)(\|u_1-u_2\|_{pq}+\|v_1-v_2\|_{pq})\\
&\leq \frac{1}{3}(\|u_1-u_2\|_{Y_T}+\|v_1-v_2\|_{Y_T}).
\end{aligned}
$$

Similarly as above we obtain $\|\Phi^2_{(u_0,v_0)}(u_1,v_1) - \Phi^2_{(u_0,v_0)}(u_2,v_2)\|_{Y_T} \le \frac{1}{3}(\|u_1-u_2\|_{Y_T} + \|v_1-v_2\|_{Y_T})$. It implies that $\|\Phi_{(u_0,v_0)}(u_1,v_1) - \Phi_{(u_0,v_0)}(u_1,v_1)\|_{Y_T} \le \frac{2}{3}\|(u_1,v_1)-(u_2,v_2)\|_{Y_T}$. Consequently, $\Phi_{(u_0,v_0)}$ is a contraction in B_M and it possesses a unique fixed point (u,v) in this set.

Note for further reference that in fact, for any $T \le T_0$,

$$(u,v) \text{ is the only solution of } \Phi_{(u_0,v_0)}(u,v) = (u,v) \text{ in } Y_T. \tag{8}$$

Step 2. Regularity. For the function $f(u,v) = \mu_1 u^p + \beta u^r v^{r+1} \in L^1((0,T), L^q(\Omega))$, hence $u = \Phi^1_{(u_0,v_0)}(u,v) \in C([0,T), L^q(\Omega) \times L^q(\Omega))$. Choose $\varepsilon > 0$ small and set $k_1 = pq$. Then

$$u \in L^\infty([\varepsilon,T), L^{k_1}(\Omega)),$$

and

$$\begin{aligned} &u(t+\varepsilon)\\ &= e^{-tA}u(\varepsilon) + \int_0^t e^{-(t-s)A} f(u(s+\varepsilon), v(s+\varepsilon))ds \\ &= e^{-tA}u(\varepsilon) + \int_0^t e^{-(t-s)A}(\mu_1 u(s+\varepsilon)^p + \beta u(s+\varepsilon)^r v(s+\varepsilon)^{r+1}). \end{aligned} \tag{9}$$

Choose $k_2 > k_1 > pk_3$ such that $\beta_1 = \frac{n}{2}(\frac{p}{k_1} - \frac{1}{k_2}) > 1$. Set $\beta_2 = \frac{n}{2}(\frac{1}{k_1} - \frac{1}{k_2})$, $\beta_3 = \frac{n}{2}(\frac{1}{k_3} - \frac{1}{k_1}) < 1$. Using (9) and the L^p-L^q-estimate and Hölder inequality, we get

$$\begin{aligned} \|u(t+\varepsilon)\|_{k_2} \le\ & \|e^{-tA}u(\varepsilon)\|_{k_2} \\ &+ \int_0^t \|e^{-(t-s)A}\mu_1 u^p(s+\varepsilon)\|_{k_2} ds + \int_0^t \|e^{-(t-s)A}\beta u^r(s+\varepsilon)v^{r+1}(s+\varepsilon)\|_{k_2} ds \\ \le\ & t^{-\beta_2}\|u(\varepsilon)\|_{k_1} + |\mu_1| \int_0^t (t-s)^{-\beta_1}\|u(s+\varepsilon)\|^p_{k_1} ds \\ &+ |\beta| \int_0^t \|e^{-(t-s)A}u^r(s+\varepsilon)v^{r+1}(s+\varepsilon)\|_{k_2} ds \\ \le\ & t^{-\beta_2}\|u(\varepsilon)\|_{k_1} + |\mu_1| \int_0^t (t-s)^{-\beta_1}\|u(s+\varepsilon)\|^p_{k_1} ds \\ &+ |\beta| \int_0^t (t-s)^{-\beta_3}\|u^r(s+\varepsilon)v^{r+1}(s+\varepsilon)\|_{k_3} ds \\ \le\ & t^{-\beta_2}\|u(\varepsilon)\|_{k_1} + |\mu_1| \int_0^t (t-s)^{-\beta_1}\|u(s+\varepsilon)\|^p_{k_1} ds + C(\Omega,k_1,k_3,r) \\ &+ |\beta| \int_0^t (t-s)^{-\beta_3}\|u(s+\varepsilon)\|^{rk_3}_{k_3}\|v(s+\varepsilon)\|^{(r+1)k_3}_{k_3} \le C(\varepsilon) \end{aligned}$$

for $t \in [\varepsilon, T-\varepsilon]$. Hence $u \in L^\infty([2\varepsilon,T], L^{k_2}(\Omega))$ and a bootstrap argument shows that $u \in L^\infty_{loc}((0,T), L^\infty(\Omega))$. Similarity, we can get $v \in L^\infty_{loc}((0,T), L^\infty(\Omega))$. Standard existence and regularity results for linear parabolic equations (see [15, Theorem48.1,Theorem 48.2]) guarantee that u is a classical solution for $t > 0$, hence a classical L^q-solution. Let we explain this in more details in the case of bounded domain. In the general case, one can use smooth cut-off functions and localized versions of the regularity statements. For bounded domain Ω, we are going to prove (u,v) is a classical solution for $t > 0$.

Fix $\delta > 0$ small and let $\psi : \mathbb{R} \to [0,1]$ be a smooth function satisfying $\psi(t) = 0$ for $t \le \delta$ and $\psi(t) = 1$ for $t \ge 2\delta$. Since (u,v) is a mild solution, from Corollary 6 we know that (u,v) is also a weak L^1_δ-solution. Consequently, $(\psi u, \psi v)$ is a weak solution of the linear problem (4) with $f := \psi_t u + \psi(|\mu_1||u|^{p-1}u + \beta|u|^{r-1}u|v|^{r+1}) \in L^\infty((0,T)\times\Omega)$, $g := \psi_t v + \psi(|\mu_2||v|^{p-1}v + \beta|u|^{r+1}|v|^{r-1}v) \in L^\infty((0,T)\times\Omega)$. Now Theorem 7.13, 7.15, 7.17 and Corollary 7.16 in [9] guarantee that this linear problem has a strong solution $(u',v') \in W^{2,1;q}((0,T),\Omega\times\Omega)$ for any $q \in (1,\infty)$. This strong solution is obviously a weak solution and from Proposition 4 we know that the uniqueness of weak solution guarantee $(\psi u, \psi v) = (u',v')$, consequently $(u,v) \in W^{2,1;q}((2\delta,T),\Omega\times\Omega)$. Now fixing $q > n+2$ we see that $f(u,v)$ is Hölder continuous in $((2\delta,T),\Omega\times\Omega)$ and $g(u,v)$ is Hölder continuous in $((2\delta,T),\Omega\times\Omega)$. Next consider the function

$(\psi(t-2\delta)u(t), \psi(t-2\delta)v(t))$, use Theorem 4.28 and Theorem 5.14 in [9] , we can see that (u,v) is a classical solution for $t > 4\delta$.

Step 3. Continuous dependence. Let us denote by $U(t)(u_0^1, v_0^1) = (U_1(t), U_2(t))(u_0^1, v_0^1)$ the solution $(u(t), v(t))$ constructed above. The existence proof shows that $U(\cdot)(u_0^2, v_0^2)$ is defined and belongs to $B_{M,T}$ for any (u_0^2, v_0^2) satisfying $\|(u_0^2, v_0^2)\|_q < M$ and any $T \le T_0$. In addition, (5) guarantees

$$\|U_1(\cdot)(u_0^1, v_0^1) - U_1(\cdot)(u_0^2, v_0^2)\|_{Y_T} \le \|u_0^1 - u_0^2\|_q + C(p,\alpha)M^{p-1}T^{1-p\alpha}(\|u_1 - u_2\|_{Y_T} + \|v_1 - v_2\|_{Y_T}),$$

hence the choice of T_0 implies $\|U_1(\cdot)(u_0^1, v_0^1) - U_1(\cdot)(u_0^2, v_0^2)\|_{Y_T} \le 2\|u_0^1 - u_0^2\|_q$. Similarity as above, we can get $\|U_2(\cdot)(u_0^1, v_0^1) - U_2(\cdot)(u_0^2, v_0^2)\|_{Y_T} \le 2\|v_0^1 - v_0^2\|_q$. Consequently, $\|U(\cdot)(u_0^1, v_0^1) - U(\cdot)(u_0^2, v_0^2)\|_{Y_T} \le 2(\|u_0^1 - u_0^2\|_q + \|v_0^1 - v_0^2\|_q)$.

It follows that

$$\begin{aligned}
&\|U_1(t)(u_0^1, v_0^1) - U_1(t)(u_0^2, v_0^2)\|_q \\
&\le \|u_0^1 - u_0^2\|_q + |\mu_1| \int_0^t \||U_1(s)(u_0^1, v_0^1)|^{p-1} U_1(s)(u_0^1, v_0^1) - |U_1(s)(u_0^2, v_0^2)|^{p-1} U_1(s)(u_0^2, v_0^2)\|_q ds \\
&\quad + |\beta| \int_0^t \||U_1(s)(u_0^1, v_0^1)|^{r-1} U_1(s)(u_0^1, v_0^1) |U_2(s)(u_0^1, v_0^1)|^r\|_q ds \\
&\quad - |\beta| \int_0^t \||U_1(s)(u_0^2, v_0^2)|^{r-1} U_1(s)(u_0^2, v_0^2) |U_2(s)(u_0^2, v_0^2)|^r\|_q ds
\end{aligned}$$

$$\begin{aligned}
&\le \|u_0^1 - u_0^2\|_q + |\mu_1| C(p) M^{p-1} T_0^{1-\alpha} \|U_1(\cdot)(u_0^1, v_0^1) - U_1(\cdot)(u_0^2, v_0^2)\|_{Y_T} \\
&\quad + |\beta| C(r) M^{p-1} T_0^{1-p\alpha} \|U_1(u_0^1, v_0^1) - U_1(u_0^2, v_0^2)\|_{Y_T} \\
&\quad + |\beta| C(r+1) M^{p-1} T_0^{1-p\alpha} \|U_2(u_0^1, v_0^1) - U_2(u_0^2, v_0^2)\|_{Y_T} \\
&\le \|u_0^1 - u_0^2\|_q + C' M^{p-1} T_0^{1-p\alpha} \|U(u_0^1, v_0^1) - U(u_0^2, v_0^2)\|_{Y_T} \\
&\le C(\|u_0^1 - u_0^2\|_q + \|v_0^1 - v_0^2\|_q).
\end{aligned}$$

Similarity as above we can get $\|U_2(t)(u_0^1, v_0^1) - U_2(t)(u_0^2, v_0^2)\|_q \le C(\|u_0^1 - u_0^2\|_q + \|v_0^1 - v_0^2\|_q)$.

Hence

$$\|U(t)(u_0^1, v_0^1) - U(t)(u_0^2, v_0^2)\|_q \le C(\|u_0^1 - u_0^2\|_q + \|v_0^1 - v_0^2\|_q). \tag{10}$$

whenever $t \le T_0$. Consequently, the map $L^q(\Omega) \times L^q(\Omega) \to L^q(\Omega) \times L^q(\Omega) : (u_0^2, v_0^2) \to U(t)(u_0^2, v_0^2)$ is Lipschitz continuous in a neighborhood of (u_0^1, v_0^1).

Step 4. Uniqueness. Let (u', v') be a classical $L^q \times L^q$-solution of (1.1) in a interval $[0, T_1)$. Due to uniqueness property (8), it is sufficient to show that $(u', v') = U(t)(u_0, v_0)$ for small t.

Step 5. Smoothing estimate. Fix $M = 2\|(u_0, v_0)\|_q$ and notice that $T_0 = T_0(\|(u_0, v_0)\|_q)$ (provided we suppress the dependence of T_0 on n, p, q). Choose $k \ge q$. If $s = q$ or $s = pq$, then (5) follows from (10) (with $(u_0^2, v_0^2) = (0, 0)$) or (7) respectively. Assume that

$$\|(u(t), v(t))\|_m \le C\|(u_0, v_0)\|_q t^{-\alpha_m} \tag{11}$$

for some $m \ge \max(p, q)$, where α_m is defined in (5). And we can easily deduce that $\|u(t)\|_m \le C\|(u_0, v_0)\|_q t^{-\alpha_m}$.

Since $m \geq q$, $p/m - 1/m < 2/n$, we have

$$\|u(t)\|_k \leq \|e^{-\frac{t}{2}A}u(\frac{t}{2})\|_k + \int_{\frac{t}{2}}^{t} \|e^{-(t-s)A}(\mu_1|u(s)|^{p-1}u(s) + \beta|u(s)|^{r-1}u(s)|v(s)|^{r+1})\|_k ds$$

$$\leq \|e^{-\frac{t}{2}A}u(\frac{t}{2})\|_k + |\mu_1|\int_{\frac{t}{2}}^{t} \|e^{-(t-s)A}|u(s)|^{p-1}u(s)\|_k$$

$$+ |\beta|\int_{\frac{t}{2}}^{t} \|e^{-(t-s)A}|u(s)|^{r-1}u(s)|v(s)|^{r+1}\|_k ds$$

$$\leq t^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{k})}\|u(\frac{t}{2})\|_m + |\mu_1|\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{k})}\|u(s)\|_m^p ds$$

$$+ |\beta|\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{k})}\|u(s)\|_m^r\|v(s)\|_m^{r+1} ds$$

$$\leq C\|(u_0,v_0)\|_q t^{-\alpha_k} + (|\mu_1|+|\beta|)\int_{\frac{t}{2}}^{t} C^p\|(u_0,v_0)\|_q^p s^{-p\alpha_m} ds$$

$$\leq C\|(u_0,v_0)\|_q t^{-\alpha_k}(1 + t^{\left(1-\frac{n(p-1)}{2q}\right)}\int_{\frac{1}{2}}^{1}(1-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{k})}s^{p(\frac{n}{2})(\frac{1}{q}-\frac{1}{m})}ds)$$

$$\leq C\|(u_0,v_0)\|_q t^{-\alpha_k}$$

provided $p/m - 1/k < 2/n$. Hence in a finite number of iterations and enlarging C if necessary, we can obtain

$$\|u(t)\|_\infty \leq C\|(u_0,v_0)\|_q t^{-\alpha_\infty}.$$

By the use of interpolation inequality

$$\|u(t)\|_k \leq \|u(t)\|_q^{\frac{q}{k}}\|u(t)\|_\infty^{1-\frac{q}{k}},$$

we can get for any $k \in [q,\infty]$

$$\|u(t)\|_k \leq C\|(u_0,v_0)\|_q t^{-\alpha_k}.$$

Similarity, we can get

$$\|v(t)\|_k \leq C\|(u_0,v_0)\|_q t^{-\alpha_k}.$$

Consequently, $\|(u(t),v(t))\|_k \leq C\|(u_0,v_0)\|_q t^{-\alpha_k}$, for any $k \in [q,\infty]$.

Step 6. Positivity. The positivity statement follows from the nonnegativity of the semigroup e^{-tA} and the construction of the solution as a limit of nonnegative iteration $(u_{k+1},v_{k+1}) = \Phi_{u_0,v_0}(u_k,v_k)$, $(u_1,v_1) \equiv (0,0)$. ■

References

[1] J. Ball, Remarks on blow-up and nonexistence theorems dor nonlinear evolution equations, Quart. J. Math. Oxford Ser. (2)28(1977),473-486.

[2] H. Brezis, T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. 68(1996),277-304.

[3] A.-T. Duong, Q.-H. Phan, A Liouville-type theorem for cooperative parabolic systems, Discret. Contin. Dyn. Syst. 38(2) (2018) 822-833.

[4] M. Escobedo, M.-A. Herrero, A uniqueness result for a semilinear reaction diffusion system, Proc. Amer. Math. Soc. 112 (1991), 175-185.

[5] M. Fila, Remarks on blow up for a nonlinear diffusion equations with a gradient term, Proc. Amer. Math. Soc. 111(1991). 759-801.

[6] M. Fila, Boundedness of global solutions of nonlinear diffusion equations, J. Differential Equations 98(1992),226-240.

[7] M. Fila, P. Souplet, F.-B. Weissler, Linear and nonlinear heat equations in L^p_δ sapces and universal bounds for global solutions, Math. Ann. 320(2001),87-113.

[8] M. Ishiwata, B. Ruf, F. Sani, E. Terraneo, A potential well argument for a semilinear parabolic equation with exponential nonlinearity, Matrix. Ann. 3(2020),265-273.

[9] G.-M. Lieberman, Second order parabolic differential equations, World Scientific, Singapore, 1996.
[10] L.-E. Panye, D.-H. Sattinger,Saddle points and instability of nonlinear hyperbolic equations, Israel J. Math. 22,273-303(1975).
[11] Q.-H. Phan, P. Souplet, A Liouville-type theorem for the 3-dimensional parabolic Gross-Pitaevskii, Math. Ann. 366(2016),1561-1585.
[12] P. Polacik, P. Quittner, P. Souplet, Singularity and decy eatimates in superlinear problems via Liouville-type theorems. Part II: prabolic equations, Indiana Univ. Math. J. 56(2007),879-908.
[13] P. Quittner, P. Souplet, Admissable L_p norm for local existence and for continuation in semilinear parabolic systems are not the same, Proc. Roy. Soc. Edingburgh 131(2001),1425-1456.
[14] P. Quittner, P. Souplet, Global existence from single-component L^p estimates in a semilinear reaction-diffusion system, Proc. Amer. Math. Soc. 130 (2002), 2719-2724.
[15] P. Quittner, P. Souplet: Superlinear Parabolic Problems: Blow-Up, Global Existence and Steady States, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] (Birkhäuser Verlag, Basel) (2007).
[16] P. Souplet, Geometry of unbounded domains, poincaré inequalities and stability in semilinear parabolic equations, Commun. Partial. Differ. Equ. Equations, 24:5-6(1999), 545-551.
[17] P. Souplet, Decay of Heat Semigroups in L^∞ and Applications to Nonlinear Parabolic Problems in Unbounded Domains, J. Funct. Anal. 173(2000), 343-360.