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Stabilization of Finitely Differentiable Linear Time-Varying Systems via Reducibility

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Abstract: In this paper, we study the stabilization of the special kind of linear time-varying systems. More precisely, we show the full measure reducibility of finitely differentiable quasi-periodic linear systems with respect to the rotation number in the local perturbative regime by the improved KAM theory, based on which we design the time-varying feedback controller to stabilize them.

Keywords: Reducibility; Time-varying; KAM theory

1 Introduction

In this paper, we study the stability of the time-varying system

$$\dot{x} = f(t, x) + u(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is called the state, $f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^k function with $k < +\infty$, and $u : \mathbb{R} \to \mathbb{R}^n$ is called the controller. We assume that f in (1) is linear with respect to the state, i.e. $f(t, x) = A(t)x(t)$, where $A \in C^k(\mathbb{R}, \mathrm{gl}(n, \mathbb{R}))$, thus (1) is a linear time-varying system with controller. The stability of controlled system has deep relation with the dynamics of the following homogeneous system

$$\dot{x} = A(t)x(t). \tag{2}$$

The cohomological equation may be solved if $A(t)$ is quasi-periodic and close to constant. Let us recall the notion of quasi-periodic functions.

Definition 1 *The matrix function $F(t) : \mathbb{R} \to \mathrm{gl}(n, \mathbb{R})$ is called quasi-periodic with frequency $\omega_1, \omega_2, \cdots, \omega_d$ if there exists $G(\theta) : \mathbb{T}^d \to \mathrm{gl}(n, \mathbb{R})$ being periodic with respect to each θ_i such that $F(t) = G(\omega_1 t, \omega_2 t, \cdots, \omega_d t)$.*

Suppose that F is defined on the d-dimensional tours $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and A is a constant matrix, then (2) is a linear quasi-periodic system:

$$\dot{x} = (A + F(\theta))x(t), \quad \dot{\theta} = \omega, \tag{3}$$

where $\omega \in \mathbb{R}^d$ is the frequency satisfying that $(1, \omega)$ is rationally independent. Simply, we abbreviate system (3) as $(\omega, A + F)$ and abbreviate the corresponding controlled system $\dot{x} = (A + F(\omega t))x(t) + u(t)$ as $(\omega, A + F, u)$.

Definition 2 *Given the system $(\omega, A(\theta))$ with $A \in C^k(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{R}))$, if there exists a transformation $x \mapsto B(\theta)x$ with $B \in C^{k_0}(2\mathbb{T}^d, \mathrm{GL}(n, \mathbb{R}))$ conjugating $(\omega, A(\theta))$ to a constant system, then $(\omega, A(\theta))$ is C^{k,k_0} reducible.*

Definition 3 *We say $(\omega, A(\theta))$ is C^{k,k_0} almost reducible if there exists a sequence of transformations $\{B_j(\theta)\}_{j=1}^{\infty} \in C^{k_0}(2\mathbb{T}^d, \mathrm{GL}(n, \mathbb{R}))$ such that the tranformated systems converge to the constant system when j goes to $+\infty$.*

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In addition to the quasi-periodic linear systems (i.e. continuous case), another aspect of reducibility comes from the quasi-periodic linear cocycles (i.e. discrete case). Let $(\alpha, Ae^{f(\theta)}) : \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{T}^d \times \mathbb{R}^n$ $((\theta, x) \mapsto (\theta + \alpha, A(\theta)x))$ be the quasi-periodic linear cocycle, where $A \in \mathrm{GL}(n, \mathbb{R})$ and $f : \mathbb{T}^d \to \mathrm{gl}(n, \mathbb{R})$. To distinguish it from continuous case, the frequency is usually written as α, not ω, and we abbreviate it as (α, Ae^f).

We say that $\omega \in \mathbb{R}^d$ is called Diophantine, denoted by $\omega \in \mathrm{DC}(\gamma, \tau)$, if there are $\gamma > 0$ and $\tau > d$ such that $\inf_{j\in\mathbb{Z}} |\langle n, \omega\rangle - j| > \frac{\gamma}{|n|^\tau}$, $\forall\, n \in \mathbb{Z}^d\backslash\{0\}$, where $\langle n, \omega\rangle = n_1\omega_1 + ... + n_d\omega_d$ and $|n| = |n_1| + \cdots + |n_d|$. Denote $\mathrm{DC} = \bigcup_{\gamma>0,\tau>d} \mathrm{DC}(\gamma, \tau)$, which is of full Lebesgue measure [1].

Based on the KAM iteration which was proposed in the analytic and continuous cases originally in [8], Cai-Chavaudret-You-Zhou [3] showed that the finitely differentiable quasi-periodic $\mathrm{SL}(2, \mathbb{R})$-value cocycles $(\alpha, Ae^{f(\theta)})$ is almost reducible if $f \in C^k(\mathbb{T}^d, \mathrm{sl}(2, \mathbb{R}))$ is small and $\alpha \in \mathrm{DC}$. Hou and You [6] studied the one-frequency quasi-periodic systems $(\omega, A + F)$ with $F \in C^\omega(\mathbb{T}^2, \mathrm{sl}(2, \mathbb{R}))$ and $\omega_1 = 1$ and $\omega_2 \in \mathbb{R}\backslash\mathbb{Q}$. They proved that there exists $\varepsilon > 0$ such that if $\sup_{|\Im\theta|<h} \|F\| < \varepsilon$, $(\omega, A + F)$ is non-perturbative almost reducibility, which means ε is independent of the frequency ω. However, if dropping the assumption of non-resonant condition, there is no result showing the reducibility of the finitely differentiable system as far as we know. Motivated by references [2], [3] and [9], this paper will deal with the two dimensional continuous systems $(\omega, A + F(\theta))$ with $A \in \mathrm{sl}(2, \mathbb{R})$ and $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2, \mathbb{R}))$. Here is the main result.

Theorem 1 *Let $\omega \in \mathrm{DC}(\gamma, \tau)$ and consider the system $(\omega, A + F(\theta))$ with $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2, \mathbb{R}))$ and $k > 5D\tau$, where D is a large numerical constant. There is a set $\mathfrak{S} \subset [0, 1]$ being of full measure and assume $\mathrm{rot}(A + F) \in \mathfrak{S}$. Then there exists $\epsilon > 0$ such that if $\|F\|_k < \epsilon$, the following hold:*

(1) there exist $B \in C^{\frac{k}{400}}(2\mathbb{T}^d, \mathrm{SL}(2, \mathbb{R}))$ and $\check{A} \in \mathrm{sl}(2, \mathbb{R})$ such that $(\omega, A + F)$ is conjugated to the constant system $(\omega, \check{A})$ by B.

(2) Moreover, if there exist constant $K \in \mathrm{gl}(2, \mathbb{R})$ and symmetric positive define matrix $Q \in \mathrm{gl}(2, \mathbb{R})$ such that $(\check{A} - K)^T Q + Q(\check{A} - K) < 0$, then the controlled system $(\omega, A + F, u)$ is asymptotically stable with controller $u(t) = -B(\omega t)KB(\omega t)^{-1}x(t)$.

2 Preliminaries

We denote the space of k times differentiable with continuous k-th derivatives matrix-valued functions defined on $\mathbb{T}^d$ by $C^k(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{C}))$ with $k \in \mathbb{Z}$, which is equipped with the norm $\|F\|_k = \sup_{|s|\le k,\theta\in\mathbb{T}^d} \|\partial^s F(\theta)\|$, where $|s| = |s_1| + \cdots + |s_d|$ and $s_i \in \mathbb{Z}$ for any $i = 1, \cdots, d$. Let $C^\omega(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{R})) = \bigcup_{h>0} C^\omega_h(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{R}))$, where $C^\omega_h(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{R}))$ is the space of bounded real analytic functions defined on $\mathbb{T}^d$ with complex extensions on the band $|\Im\theta| < h$ with $|\Im\theta| = |\Im\theta_1| + \cdots + |\Im\theta_d|$. The norm in $C^\omega_h(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{R}))$ is defined as $|F|_h = \sup_{\theta\in\mathbb{T}^d,|\Im\theta|<h} \|F(\theta)\|$. If $\alpha \in \mathbb{R}$, let $\|\alpha\|_{\mathbb{T}} = \min_{l\in\mathbb{Z}} |\alpha - l|$.

Definition 4 *For any $F(\theta) : \mathbb{T}^d \to \mathrm{gl}(n, \mathbb{R})$, let $F(\theta) = \sum_{k\in\mathbb{Z}^d} \widehat{F}(k)e^{2\pi i\langle k,\theta\rangle}$, $\widehat{F}(k) = \oint_{\mathbb{T}^d} F(\theta)e^{-2\pi i\langle k,\theta\rangle}d\theta$ be the Fourier expansion. For any $N > 0$, the truncation operators $\mathcal{T}_N$ and $\mathcal{R}_N$ are defined as*

$$\mathcal{T}_N F(\theta) = \sum_{k\in\mathbb{Z}^d,|k|\le N} \widehat{F}(k)e^{2\pi i\langle k,\theta\rangle},$$
$$\mathcal{R}_N F(\theta) = \sum_{k\in\mathbb{Z}^d,|k|>N} \widehat{F}(k)e^{2\pi i\langle k,\theta\rangle}.$$

If $F(\theta) \in C^\omega_h(\mathbb{T}^d, \mathrm{gl}(n, \mathbb{R}))$, one can show the exponential decay of the Fourier coefficient $\widehat{F}(k)$:

$$\|\widehat{F}(k)\| \le |F(\theta)|_h e^{-2\pi|k|h}. \tag{4}$$

By a simple calculation, the following estimate holds:

$$|\mathcal{R}_N F(\theta)|_{h'} \le |F(\theta)|_h e^{-2\pi N(h-h')},\ \ \forall\, 0 < h' < h. \tag{5}$$

Definition 5 *Given any $\omega \in \mathbb{R}^d$, for $\eta > 0$ and $A \in \mathrm{sl}(2, \mathbb{R})$, one can decompose $\mathfrak{B}_h = \mathfrak{B}_h^{(nre)}(\eta) \oplus \mathfrak{B}_h^{(re)}(\eta)$ such that for any $Y \in \mathfrak{B}_h^{(nre)}(\eta)$, $\partial_\omega Y$, $[A, Y] \in \mathfrak{B}_h^{(nre)}(\eta)$, and $|\partial_\omega Y - [A, Y]|_h \ge \eta|Y|_h$, where $\partial_\omega Y = \langle\omega, \frac{\partial Y}{\partial\theta}\rangle = \frac{dY}{dt}$ is the direction derivative of Y along with ω and $[\cdot, \cdot]$ is Lie bracket: $[A, B] = AB - BA$.*

Lemma 2 ([6]) *Given $F \in \mathfrak{B}_h$ with $|F|_h < \varepsilon$, assume that $\varepsilon \in (0, 10^{-8})$ and $\eta \geq \varepsilon^{\frac{1}{4}}$. There exist $Y \in \mathfrak{B}_h, G \in \mathfrak{B}_h^{(re)}(\eta)$ with the estimates $|Y|_h \leq \varepsilon^{\frac{1}{2}}$, $|G|_h \leq 2\varepsilon$ such that $(\omega, A+F)$ is conjugated to $(\omega, A+G)$ by $e^{Y(\theta)}$.*

Lemma 3 *Let $A(t) \in C^1(\mathbb{R}, \mathrm{GL}(n,\mathbb{C}))$, the derivative of $A(t)^{-1}$ is $\frac{d}{dt}(A(t)^{-1}) = -A(t)^{-1}\frac{d}{dt}A(t)(A(t)^{-1})$.*

Lemma 4 (Schur complement) *For given the symmetric matrix $Q = \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix}$ with $Q_{i,j} \in \mathbb{R}, i,j = 1,2$, the following two conditions are equivalent:*

(1) $Q < 0$;

(2) $Q_{1,1} < 0$, and $Q_{2,2} - \frac{Q_{1,2}^2}{Q_{1,1}} < 0$.

Definition 6 *For any $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$, there exist $\{F_j\}_{j\geq 1} \in C^{\omega}_{\frac{1}{j}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ and a universal constant $C' > 0$ such that[11]*

$$
\begin{aligned}
&\|F_j - F\|_k \to 0, \;\; j \to +\infty, \\
&|F_{j+1} - F_j|_{\frac{1}{j+1}} \leq C'(j)^{-k}\|F\|_k, \;\; \forall\, j \in \mathbb{Z}^+, \\
&|F_j|_{\frac{1}{j}} \leq C'\|F\|_k, \;\; \forall\, j \in \mathbb{Z}^+.
\end{aligned} \tag{6}
$$

Definition 7 *Assume that $\Phi(\theta_0, t)$ is the basic solution of the system $(\omega, A(\theta))$ with $A \in C^1(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$, the rotation number of the system is defined $\mathrm{rot}(A) = \lim_{t\to+\infty}\frac{\arg[\Phi(\theta_0,t)x_0]}{2\pi t}$, where $x_0 \in \mathbb{R}^2\backslash\{0\}$ and $\theta_0 \in \mathbb{T}^d$.*

The rotation number $\mathrm{rot}(A)$ takes value in $[0,1]$ and does not depend on θ_0 and x_0, see [7]. Recall that $\mathrm{rot}(A)$ is Diophantine with respect to ω if there exist $\kappa > 0, \tau > d$ such that the following holds:

$$
\inf_{j\in\mathbb{Z}} |2\mathrm{rot}(A) - \langle n, \omega\rangle - j| > \frac{\kappa}{|n|^{\tau}}, \;\; \forall\, n \in \mathbb{Z}^d\backslash\{0\},
$$

and we denote it by $\mathrm{rot}(A) \in \mathrm{DC}_{\omega}(\kappa, \tau)$. The rotation number $\mathrm{rot}(A)$ is said to be rational with respect to ω if $2\mathrm{rot}(A) = \langle m, \omega\rangle \mod \mathbb{Z}$ for some $m \in \mathbb{Z}^d$. For simplicity, we omit "$\mathrm{mod}\ \mathbb{Z}$" in the rest of the paper.

Define $R_{\phi} = \begin{pmatrix} \cos 2\pi\phi & -\sin 2\pi\phi \\ \sin 2\pi\phi & \cos 2\pi\phi \end{pmatrix}$, and suppose that $A \in C^0(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$ is homotopic to $R_{\frac{\langle n,\theta\rangle}{2}}$ for some $n \in \mathbb{Z}^d$, then we call n the degree of A and denote it by $\deg A$. Moreover,

$$
\deg(AB) = \deg A + \deg B. \tag{7}
$$

If (ω, A_1) is conjugated to (ω, A_2) by $B \in C^1(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$, then

$$
\mathrm{rot}(A_2) = \mathrm{rot}(A_1) - \frac{\langle \deg B, \omega\rangle}{2}. \tag{8}
$$

If (ω, A_1) is close to (ω, A_2) enough in C^0 norm, then by the definition of rotation number, one can get that there exists constant $\tilde{C}$ such that

$$
|\mathrm{rot}(A_1) - \mathrm{rot}(A_2)| \leq \tilde{C}\|A_1 - A_2\|_0^{\frac{1}{2}}. \tag{9}
$$

3 Design of the controller

Consider the following quasi-periodic linear system

$$
\begin{cases} \dot{x} = (A + F(\theta))x \\ \dot{\theta} = \omega \end{cases}, \tag{10}
$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{T}^d$, $A \in \mathrm{gl}(n,\mathbb{R})$, and $F \in C^1(\mathbb{T}^d, \mathrm{gl}(n,\mathbb{R}))$. One can obtain the asymptotical stability of the following system via a special designed controller $u(t)$, i.e.,

$$
\begin{cases} \dot{x} = (A + F(\theta))x + u(t) \\ \dot{\theta} = \omega \end{cases}. \tag{11}
$$

Again, we abbreviate the systems (10) and (11) as $(\omega, A+F)$ and $(\omega, A+F, u)$ respectively.

Proposition 5 *If there exist $B \in C^1(2\mathbb{T}^d, \mathrm{GL}(n,\mathbb{R}))$ and $\check{A} \in \mathrm{gl}(n,\mathbb{R})$ to solve the following cohomological equation*

$$\partial_\omega B(\theta) = (A + F(\theta))B(\theta) - B(\theta)\check{A}, \tag{12}$$

then the state of system (11) is asymptotically stable via $u(t) = B(\omega t)v(t)$, whenever the state of $(\omega, \check{A}, v)$ is asymptotically stable, where $v \in C^0(\mathbb{R}, \mathbb{R}^n)$.

Now we can choose the suitable controller $u(t)$ in (11).

Theorem 6 *Suppose that the condition in Proposition 5 holds. If there exist symmetric positive definite matrix $Q \in \mathrm{gl}(n,\mathbb{R})$ and $K \in \mathrm{gl}(n,\mathbb{R})$ satisfying that*

$$\Sigma \triangleq (\check{A} - K)^T Q + Q(\check{A} - K) < 0, \tag{13}$$

then the state of the system (11) is asymptotically stable via $u(t) = -B(\omega t)KB(\omega t)^{-1}x(t)$, where $B \in C^1(2\mathbb{T}^d, \mathrm{GL}(n,\mathbb{R}))$ and $\check{A} \in \mathrm{gl}(n,\mathbb{R})$ are defined in Proposition 5.

Remark 7 *Theorem 6 provides an sufficient criterion to check the stabilization of the system (11). In fact, one can get the equivalent criterion that $\check{A} - K < 0$ as a corollary of Proposition 5.*

Remark 8 *The controller $u(t)$ in Theorem 6 is the time-varying linear feedback controller, however it can be extended to the other situation via different controllers obviously. Here we only consider the linear feedback control.*

4 Reducibility by KAM argument

In this part, we are going to show that if the non-constant part $F(\theta)$ is sufficiently small, then the C^k quasi-periodic linear system $(\omega, A + F(\theta))$ can be conjugated to a constant system by the improved KAM theory.

4.1 Analytic case

Consider the analytic quasi-periodic system $(\omega, A + F)$ with $A \in \mathrm{sl}(2,\mathbb{R})$ is constant and $F(\theta) \in C_h^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ is a pertubation. Then we have the following results.

Proposition 9 *[3, 8] Let $\omega \in \mathrm{DC}(\gamma,\tau), h \in (0,1)$ and $h_+ \in (0,h)$. Then there exist $c = c(\gamma,\tau,d)$ and a numerical constant D such that if*

$$|F|_h < \varepsilon < \frac{c}{\|A\|^D}(h - h_+)^{D\tau}, \tag{14}$$

then there exist $B \in C_{h_+}^\omega(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$, $A_+ \in \mathrm{sl}(2,\mathbb{R})$ and $F_+ \in C_{h_+}^\omega(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ such that the system $(\omega, A+F(\theta))$ is conjugated to $(\omega, A_+ + F_+(\theta))$ by B, i.e., $\partial_\omega B = (A + F(\theta))B - B(A_+ + F_+(\theta))$.

Remark 10 *Even though Proposition 9 has been proved in [8] essentially and it also be proved in discrete version in [3], a brief proof is given since in engineering the design of controller depends on the construction of the transformation B.*

To get the quantitative estimates of B, A_+ and F_+, we need distinguish non-resonant case and resonant case. Let $N \triangleq \frac{2|\ln\varepsilon|}{h-h_+}$ and $\pm 2\pi i\xi$ be the two eigenvalues of A with $\xi \in \mathbb{R} \bigcup i\mathbb{R}$.

When it is non-resonant case, let $\sigma = \frac{1}{10}$, $\eta = \varepsilon^{3\sigma}$. Assume that

$$\|2\xi \pm \langle n, \omega\rangle\|_{\mathbb{T}} \geq \varepsilon^\sigma, \ \ \forall\, n \in \mathbb{Z}^d \ \ with \ \ 0 < |n| \leq N, \tag{15}$$

then we have estimates: $|F_+|_{h_+} < \varepsilon^{3-2\sigma}$, $\|A_+ - A\| < 4\varepsilon$, $|B - \mathrm{Id}|_{h_+} < 2\varepsilon^{\frac{1}{2}}$, $\deg B = 0$.

When it is non-resonant case, let $\sigma = \frac{1}{10}$. If there exists $n^* \in \mathbb{Z}^d$ with $0 < |n^*| \leq N$ such that

$$\|2\xi \pm \langle n^*, \omega\rangle\|_{\mathbb{T}} < \varepsilon^\sigma, \tag{16}$$

then there is a constant $C_1 = C_1(\|A\|, \gamma) > 0$ and we have $|F_+|_{h_+} \leq \varepsilon^{100}$, $|B|_{h_+} \leq C_1 N^{\frac{\tau}{2}} \varepsilon^{\frac{-2\pi h_+}{h-h_+}}$, $\|A_+\| \leq 4\varepsilon^\sigma$, $\deg B = n^*$.

The Proposition 9 focuses on the analytic quasi-periodic linear system. In the following, we are going to design a C^k version proposition for the finitely differentiable case.

4.2 Finitely diferentiable case

Let $F(\theta) \in C^k(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$, then by (6), there exists an analytic sequence $\{F_j\}_{j\geq 1}, F_j \in C^{\omega}_{\frac{1}{j}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ which is approximating F. To make the iteration works, we need choose the suitable subsequence $\{F_{l_j}\}$. We first recall some notation given in [3, 4]. Denote

$$\varepsilon_0'(h,h') \triangleq \frac{c}{(2\|A\|)^D}(h-h')^{D\tau}, \tag{17}$$

$$\varepsilon_m \triangleq \frac{c}{(2\|A\|)^D m^{\frac{k}{4}}}, \quad N_m \triangleq \frac{2|\ln \varepsilon_m|}{\frac{1}{m}-\frac{1}{m^2}}, \quad \forall\, m \in \mathbb{Z}^+, \tag{18}$$

where D is the numerical constant. Let $M \in \mathbb{Z}^+$ and $M > \max\{100, \frac{(2\|A\|)^D}{c}\}$. Denote $l_j = M^{2^{j-1}}, j \geq 1$.

Proposition 11 *Let $\omega \in \mathrm{DC}(\gamma,\tau)$, and $A \in \mathrm{sl}(2,\mathbb{R})$, $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with $k \geq 5D\tau$. Let $\{F_{l_j}\}_{j\geq 1}$ is the analytic subsequence approximating $F(\theta)$, there exists ϵ_1 such that if*

$$\|F\|_k < \epsilon_1 < \frac{c}{C'(2\|A\|)^D M^{\frac{k}{4}}}, \tag{19}$$

where C' is defined in (6) and c is defined in Proposition 9, then there exist $B_{l_j} \in C^{\omega}_{\frac{1}{l_{j+1}}}(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$, $A_{l_j} \in \mathrm{sl}(2,\mathbb{R})$ and $F'_{l_j} \in C^{\omega}_{\frac{1}{l_{j+1}}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ such that $\partial_\omega B_{l_j}(\theta) = (A + F_{l_j}(\theta))B_{l_j} - B_{l_j}(A_{l_j} + F'_{l_j}(\theta))$. Moreover, let $\varrho = \frac{2\pi}{M-1}$, the following estimates hold

$$|B_{l_j}|_{\frac{1}{l_{j+1}}} \leq (l_j|\ln \varepsilon_{l_j}|)^{\tau(1+\sigma)}\varepsilon_{l_j}^{-\varrho}, \quad |F'_{l_j}|_{\frac{1}{l_{j+1}}} \leq \frac{1}{2}\varepsilon_{l_j}^{\frac{5}{2}}, \quad \|A_{l_j}\| \leq \|A\| + 2. \tag{20}$$

4.3 Almost reducibility of the C^k quasi-periodic systems

In this section, we show the almost reducibility for the C^k quasi-perodic linear systems in the local perturbatice regime. For simplicity, we will use the notation $X \lesssim Y$ which denotes $X \leq CY$ for some constants $C > 0$.

Theorem 12 *Let $\omega \in \mathrm{DC}(\gamma,\tau)$, $A \in \mathrm{sl}(2,\mathbb{R})$, $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with $k \geq 5D\tau$, there exists $\epsilon_2 = \epsilon_2(\gamma,\tau,d,k,\|A\|)$ such that if*

$$\|F\|_k < \epsilon_2 < \frac{c}{C'(2\|A\|)^D M^{\frac{k}{4}}}, \tag{21}$$

then $(\omega, A+F(\theta))$ is C^{k,k_0} almost reducible with $k_0 = [\frac{k}{20}]$. More precisely, there exist sequences $B_{l_j} \in C^{\omega}_{\frac{1}{l_{j+1}}}(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$, $A_{l_j} \in \mathrm{sl}(2,\mathbb{R})$, $\overline{F}_{l_j} \in C^{k_0}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ such that $\partial_\omega B_{l_j}(\theta) = (A+F(\theta))B_{l_j} - B_{l_j}(A_{l_j} + \overline{F}_{l_j}(\theta))$ with $\lim_{j\to+\infty} \|\overline{F}_{l_j}\|_{k_0} = 0$.

Proof. By (21) and Proposition 11, for any l_j, one can get that $\partial_\omega B_{l_j}(\theta) = (A + F_{l_j}(\theta))B_{l_j} - B_{l_j}(A_{l_j} + F'_{l_j}(\theta))$, where $F_{l_j} \in C^{\omega}_{\frac{1}{l_{j+1}}}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ is defined in Propositon 11, then we have the following cohomological equation

$$\partial_\omega B_{l_j}(\theta) = (A + F(\theta))B_{l_j} - B_{l_j}(A_{l_j} + F'_{l_j}(\theta)) + (F_{l_j}(\theta) - F(\theta))B_{l_j}.$$

Denote $A_{l_j} + \overline{F}_{l_j}(\theta) = A_{l_j} + F'_{l_j}(\theta) - B_{l_j}^{-1}(F_{l_j} - F)B_{l_j}$. Next we will show the quantitative estimate of $\overline{F}_{l_j}(\theta)$ in C^{k_0} norm. According to Cauchy estimate and (6), for $k_0 = [\frac{k}{20}]$, we have

$$\|F_{l_j} - F_{l_{j+1}}\|_{k_0} = \sup_{|s|\leq k_0, \theta\in\mathbb{T}^d} \|\partial_{\theta_1}^{s_1}\cdots\partial_{\theta_d}^{s_d}(F_{l_j} - F_{l_{j+1}})\| \leq (l_{j+1})^{k_0}(k_0)!|F_{l_j} - F_{l_{j+1}}|_{\frac{1}{l_{j+1}}} \lesssim l_j^{-(k-2k_0)},$$

by the above inequality, one can get

$$\|F(\theta) - F_{l_j}(\theta)\|_{k_0} \leq \sum_{i=j}^{\infty} \|F_{l_{i+1}} - F_{l_i}\|_{k_0} \lesssim l_j^{-(k-2k_0-1)}. \tag{22}$$

By a simple calculation and $|F'_{l_j}|_{\frac{1}{l_{j+1}}} \le \frac{1}{2}\varepsilon_{l_j}^{\frac{5}{2}}$, we get that

$$\|F'_{l_j}(\theta)\|_{k_0} \le (l_{j+1})^{k_0}(k_0)!|F'_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \lesssim (l_j)^{2k_0}(k_0)!\varepsilon_{l_j}^{\frac{5}{2}} \lesssim l_j^{-(\frac{5k}{8}-2k_0)}. \tag{23}$$

According to $|B_{l_j}|_{\frac{1}{l_{j+1}}} \le (l_j|\ln\varepsilon_{l_j}|)^{\tau(1+\sigma)}\varepsilon_{l_j}^{-\varrho} \lesssim l_j^{\frac{k}{100}}$, we also have

$$\|B_{l_j}(\theta)\|_{k_0} \le (l_{j+1})^{k_0}(k_0)!|B_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \lesssim l_j^{\frac{k}{100}+2k_0}. \tag{24}$$

Finally, we obtain the estimate for $\overline{F}_{l_j}(\theta)$ by (22), (23) and (24),

$$\|\overline{F}_{l_j}(\theta)\|_{k_0} \le \|F'_{l_j}(\theta)\|_{k_0} + \|B_{l_j}^{-1}(F_{l_j}(\theta)-F(\theta))B_{l_j}\|_{k_0} \lesssim l_j^{-(\frac{5k}{8}-2k_0)} + l_j^{\frac{k}{50}+4k_0} \times l_j^{-(k-2k_0-1)} < \varepsilon_{l_j}^2.$$

The last inequality uses the fact $k_0 = [\frac{k}{20}]$, and we can show that in C^{k_0} topology, $\|\overline{F}_{l_j}(\theta)\|_{k_0} \to 0$ $(j \to +\infty)$, i.e., the quasi-periodic linear system $(\omega, A+F(\theta))$ is C^{k,k_0} almost reducible. ■

4.4 Full measure reducibility for the quasi-periodic systems

It is well known that the number which is Diophantine or rational with respect to the frequency ω, denote by $\mathfrak{S}$, is of full Lebesgue measure. We are going to show the reducibility for the finitely differentiable quasi-periodic linear system $(\omega, A+F)$ for rotation number $\mathrm{rot}(A+F) \in \mathfrak{S}$. If the rotation number is Diophantine, one can show the all the steps being non-resonant case during the KAM iteration, hence the composition of all conjugations is convergent. If the rotation number is rational with respect to the frequency, one can show that the number of resonances is finite.

Lemma 13 *For the quasi-periodic linear system $(\omega, A+F)$ with $\omega \in \mathrm{DC}(\gamma,\tau)$, $A \in \mathrm{sl}(2,\mathbb{R})$ and $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with $k \ge 5D\tau$, where $\tau > d-1$ and $\gamma > 0$, if the rotation number of $(\omega, A+F)$ satisfies that $\mathrm{rot}(A+F) \in \mathrm{DC}_\omega(\kappa,\tau)$ with $\kappa > 0$ or $\mathrm{rot}(A+F) = 0$, then there exists $\epsilon_3 = \epsilon_3(\gamma,\kappa,\tau,d,k,\|A\|)$ such that if*

$$\|F\|_k < \epsilon_3 < C_6\varepsilon'_0(\frac{1}{l_1}, \frac{1}{l_2}), \tag{25}$$

then $(\omega, A+F)$ is C^{k,k_1} reducible with $k_1 = [\frac{k}{20}]$, where $C_6 = C_6(\tau,\gamma,\kappa,\|A\|)$.

Remark 14 *Lemma 13 has been proved in discrete case, i.e., the quasi-periodic cocycle $(\omega, Ae^{f(\theta)})$, see Lemma 3.1 and Lemma 3.2 in [2]. In fact, the reducibility of the linear system is equivalent to the reducibility of the Poincaré cocycle according to the embedding theorem in [10].*

Theorem 15 *For the system $(\omega, A+F)$ with $\omega \in \mathrm{DC}(\gamma,\tau)$, $A \in \mathrm{sl}(2,\mathbb{R})$ and $F \in C^k(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with $k \ge 5D\tau$, where $\gamma > 0$ and $\tau > d$. There exists $\epsilon_4 = \epsilon_4(\gamma,\tau,d,k,\|A\|) > 0$ such that if*

$$\|F\|_k < \epsilon_4 < \frac{c}{C'(2\|A\|)^D M^{\frac{k}{4}}}, \tag{26}$$

then the system $(\omega, A+F)$ is C^{k,k_1} reducible with $k_1 = [\frac{k}{400}]$ whenever $\mathrm{rot}(A+F)$ is Diophantine with respect to ω or $\mathrm{rot}(A+F)$ is rational with respect to ω.

Proof. By (26) and Theorem 12, there exist $B_{l_j}(\theta) \in C^\omega_{\frac{1}{l_{j+1}}}(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$, $A_{l_j} \in \mathrm{sl}(2,\mathbb{R})$ and $\overline{F}_{l_j}(\theta) \in C^{k_0}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$ with $k_0 = [\frac{k}{20}]$ such that $\partial_\omega B_{l_j}(\theta) = (A+F(\theta))B_{l_j} - B_{l_j}(A_{l_j}+\overline{F}_{l_j}(\theta))$, with estimates $|B_{l_j}|_{\frac{1}{l_{j+1}}} \le (l_j|\ln\varepsilon_{l_j}|)^{\tau(1+\sigma)}\varepsilon_{l_j}^{-\varrho}$, $\|\overline{F}_{l_j}\|_{k_0} < \varepsilon_{l_j}^2$, $\|A_{l_j}\| \le \|A\|+2$. In the following, we devide it into two cases.

(Diophantine case) By (8) and $\mathrm{rot}(A+F) \in \mathrm{DC}_\omega(\kappa,\tau)$, for any $n \in \mathbb{Z}^d\backslash\{0\}$, we have

$$\|2\mathrm{rot}(A_{l_j}+\overline{F}_{l_j}) - \langle n,\omega\rangle\|_{\mathbb{T}} = \|2\mathrm{rot}(A+F) - \langle \deg B_{l_j},\omega\rangle - \langle n,\omega\rangle\|_{\mathbb{T}} \ge \frac{\kappa(|\deg B_{l_j}|+1)^{-\tau}}{|n|^\tau},$$

which means that $\mathrm{rot}(A_{l_j}+\overline{F}_{l_j})\in \mathrm{DC}_\omega(\kappa(|\deg B_{l_j}|+1)^{-\tau},\tau)$. From Proposition 11, $\deg \widetilde{B}_{l_j}\in\{0,n^*_{l_j}\}$ with $0<|n^*_{l_j}|\le N_{l_j}$ is got, hence

$$|\deg B_{l_j}|=|\deg\prod_{i=1}^{j}\widetilde{B}_{l_i}|\le\sum_{i=1}^{j}|\deg\widetilde{B}_{l_j}|\le 4l_j|\ln\varepsilon_{l_j}|, \tag{27}$$

and thus $\|2\mathrm{rot}(A_{l_j}+\overline{F}_{l_j})-\langle n,\omega\rangle\|_{\mathbb{T}}>\frac{\kappa^*_{l_j}}{|n|^\tau}$ for any $n\in\mathbb{Z}\backslash\{0\}$ where $\kappa^*_{l_j}=\kappa(4l_j|\ln\varepsilon_{l_j}|)^{-\tau}$.

Fix j be the smallest integer such that $\varepsilon^2_{l_j}<C_1(\tau,\gamma,\kappa^*_{l_j},\|A\|)\varepsilon'_0(\frac{1}{l_1},\frac{1}{l_2})$, and apply Lemma 13 to the quasi-periodic linear system $(\omega,A_{l_j}+\overline{F}_{l_j})$, which means there exist $\check{B}(\theta)\in C^{k_1}(2\mathbb{T}^d,\mathrm{SL}(2,\mathbb{R}))$ with $k_1=[\frac{k}{400}]$ and $\check{A}\in\mathrm{sl}(2,\mathbb{R})$ such that $\partial_\omega\check{B}=(A_{l_j}+\overline{F}_{l_j})\check{B}-\check{B}\check{A}$ with $\|\check{B}\|_{k_1}\le 2$. Define $B(\theta)\triangleq B_{l_j}(\theta)\circ\check{B}(\theta)$, by using Cauchy estimate and the relation $\varepsilon^2_{l_{j-1}}\ge C_1\varepsilon'_0(\frac{1}{l_1},\frac{1}{l_2})$, it follows that $\|B\|_{k_1}\le 2(l_{j+1})^{k_1}(k_1)!(l_j|\ln\varepsilon_{l_j}|)^{\tau(1+\sigma)}\varepsilon^{-\varrho}_{l_j}<C_2(\kappa,\gamma,\tau,d,k,\|A\|)$, where the last inequality comes from the chosen of j, which depends on $\kappa,\tau,\|A\|$, hence l_j is bounded. Finally, $(\omega,A+F)$ is conjugated to $(\omega,\check{A})$ by B, which finishes the proof for Diohantine case.

(Rational case) By (8) and $\mathrm{rot}(A+F)=\frac{\langle m,\omega\rangle}{2}$ for $m\in\mathbb{Z}^d$, we have

$$\mathrm{rot}(A_{l_j}+\overline{F}_{l_j})=\mathrm{rot}(A+F)-\frac{\langle\deg B_{l_j},\omega\rangle}{2}=\frac{\langle m-\deg B_{l_j},\omega\rangle}{2}. \tag{28}$$

If $m-\deg B_{l_j}\neq 0$, we choose $J_1=J_1(\gamma,\tau)\in\mathbb{Z}^+$ sufficient large such that for any $j\ge J_1$,

$$\varepsilon_{l_j}<\frac{\gamma\tilde{C}^{-1}}{4(|m|+4l_j|\ln\varepsilon_{l_j}|)^\tau}. \tag{29}$$

By (9), (27), (28), (29) and $\omega\in\mathrm{DC}(\gamma,\tau)$, one can get that $|\mathrm{rot}(A_{l_j})|\ge|\mathrm{rot}(A_{l_j}+\overline{F}_{l_j})|-\tilde{C}\varepsilon_{l_j}\ge\frac{\gamma}{4|m-\deg B_{l_j}|^\tau}$. By lemma 8.1 in [6], we can find $S\in\mathrm{SL}(2,\mathbb{R})$ such that $S^{-1}A_{l_j}S=\begin{pmatrix}0 & 2\pi\mathrm{rot}(A_{l_j})\\ -2\pi\mathrm{rot}(A_{l_j}) & 0\end{pmatrix}\triangleq\mathcal{A}_{l_j}$, with the estimate $\|S\|\le\sqrt{\frac{\|A_{l_j}\|}{2\pi\mathrm{rot}(A_{l_j})}}\le C_4|m-\deg B_{l_j}|^{\frac{\tau}{2}}$, where $C_4=C_4(\gamma,\tau,d,\|A\|)$. Denote $\mathcal{F}_{l_j}=S^{-1}\overline{F}_{l_j}S\in C^{k_0}(\mathbb{T}^d,\mathrm{sl}(2,\mathbb{R}))$, to kill the rotation number for the system $(\omega,\mathcal{A}_{l_j}+\mathcal{F}_{l_j})$, we define the rotation conjugation $Q(\theta)=M^{-1}\begin{pmatrix}e^{-\pi i\langle m-\deg B_{l_j},\theta\rangle} & 0\\ 0 & e^{\pi i\langle m-\deg B_{l_j},\theta\rangle}\end{pmatrix}M\in\mathrm{SO}(2,\mathbb{R})$, which gives $\partial_\omega Q=(\mathcal{A}_{l_j}+\mathcal{F}_{l_j})Q-Q(\widetilde{\mathcal{A}}_{l_j}+\widetilde{\mathcal{F}}_{l_j})$, where $\widetilde{\mathcal{A}_{l_j}}=Q^{-1}\mathcal{A}_{l_j}Q-Q^{-1}\partial_\omega Q$ with $\|\widetilde{\mathcal{A}}_{l_j}\|\le 2$ and also $\mathrm{rot}(\widetilde{\mathcal{A}}_{l_j}+\widetilde{\mathcal{F}}_{l_j})=\mathrm{rot}(\mathcal{A}_{l_j}+\mathcal{F}_{l_j})-\frac{\langle m-\deg B_{l_j},\omega\rangle}{2}=0$.

Pick $J_2=J_2(\gamma,\tau,k,d,\|A\|)>J_1$ sufficient large such that for any $j>J_2$, $C_5(|m|+4l_j|\ln\varepsilon_{l_j}|)|^{\tau+k_0}\varepsilon^2_{l_j}<C_6\varepsilon'_0(\frac{1}{l_1},\frac{1}{l_2})$. Pick $j>J_2$ and let j be fixed. Thus by using Lemma 13, one can find $\check{B}\in C^{k_1}(2\mathbb{T}^d,\mathrm{SL}(2,\mathbb{R}))$ with $k_1=[\frac{k}{400}]$ and $\check{A}\in\mathrm{sl}(2,\mathbb{R})$ such that $\partial_\omega\check{B}=(\widetilde{\mathcal{A}}_{l_j}+\widetilde{\mathcal{F}}_{l_j})\check{B}-\check{B}\check{A}$ with $\|\check{B}\|_{k_1}\le 2$. Denote $B=B_{l_j}\circ S\circ Q\circ\check{B}\in C^{k_1}(2\mathbb{T}^d,\mathrm{SL}(2,\mathbb{R}))$, we have $\|B\|_{k_1}\le C_5(\gamma,\tau,k,d,\|A\|)$, thus the system $(\omega,A+F)$ is C^{k,k_1} reducible.

If $m-\deg B_{l_j}=0$, we fix $j=j(\tau,\gamma)$ be the smallest integer such that $\varepsilon^2_{l_j}<C_6\varepsilon'_0(\frac{1}{l_1},\frac{1}{l_2})$, hence by Lemma 13, there exist $\check{B}(\theta)\in C^{k_1}(2\mathbb{T}^d,\mathrm{SL}(2,\mathbb{R}))$ with $k_1=[\frac{k}{400}]$ and $\check{A}\in\mathrm{sl}(2,\mathbb{R})$ such that $\partial_\omega\check{B}=(A_{l_j}+\overline{F}_{l_j})\check{B}-\check{B}\check{A}$ with $\|\check{B}\|_{k_1}\le 2$. Define $B=B_{l_j}\circ\check{B}\in C^{k_1}(2\mathbb{T}^d,\mathrm{SL}(2,\mathbb{R}))$, the system $(\omega,A+F)$ is C^{k,k_1} reducible via B with $\|B\|_{k_1}\le C_7(\gamma,\tau,d,k,\|A\|)$. ■

4.5 Proof of the main theorem

Proof of Theorem 1. From the definition of set $\mathfrak{S}$, the proof of Theorem 1 is obvious. Here we restate the result and call it a corollary, since the proof requires just a combination of Theorem 6 with Theorem 15.

Corollary 16 *There is a full measure subset* $\mathfrak{S}\in[0,1]$*, for the quasi-periodic linear system* $(\omega,A+F)$ *whose rotation number* $\mathrm{rot}(A+F)\in\mathfrak{S}$ *with* $\omega\in\mathrm{DC}(\gamma,\tau)$ *and* $A\in\mathrm{sl}(2,\mathbb{R}),F\in C^k(\mathbb{T}^d,\mathrm{sl}(2,\mathbb{R}))$ *with* $k>5D\tau$*, if* $\|F\|_k<\epsilon^*<\frac{c}{C'(2\|A\|)^D M^{\frac{k}{4}}}$*, then the following controlled system is asymptotically stable:*

$$\begin{cases}\dot{x}=(A+F(\theta))x+u(t)\\ \dot{\theta}=\omega\end{cases}.$$

More precisely, there exist $B \in C^{\frac{k}{400}}(2\mathbb{T}^d, \mathrm{SL}(2,\mathbb{R}))$ *and constant* $\check{A} \in \mathrm{sl}(2,\mathbb{R})$*, such that* $\partial_\omega B = (A+F)B - B\check{A}$*. And the controller* $u(t) = -B(\omega t)KB(\omega t)^{-1}x$*, where* $K \in \mathrm{gl}(2,\mathbb{R})$ *satisfies* $(\check{A}-K)^T Q + Q(\check{A}-K) < 0$ *for some constant symmetric positive define matrix* $Q \in \mathrm{gl}(2,\mathbb{R})$.

■

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References

[1] V. Arnold; *Geometrical methods in the theory of ordinary differential equations*. Berlin-Heidelberg, New York, Springer-Verlag (1983).

[2] A. Cai, L. Ge; *Reducibility of finitely differentiable quasi-periodic cocycles and its spectral applications*, arXiv:1712.09041.

[3] A. Cai, C. Chavaudret, J. You, Q. Zhou; *Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles*, Math. Z., **291(3)**, 931-958 (2019).

[4] C. Chavaudret; *Almost reducibility for finitely differentiable* $\mathrm{SL}(2,\mathbb{R})$ *valued quasi-periodic cocycles*, Nonlinearity, **25**, 481-494 (2012).

[5] H. He, J. You; *An improved result for positive measure reducibility of quasi-periodic linear systems*, Acta. Math. Sin., **22**, 77-86 (2006).

[6] X. Hou, J. You; *Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems*, Invent. Math., **190**, 209-260 (2012).

[7] R. Johnson, J. Moser; *The rotation number for almost periodic potentials*, Commun. Math. Phys., **84**, 403-438 (1982).

[8] M. Leguil, J. You, Z. Zhao, Q. Zhou; *Asymptotics of spectral gaps of quasi-periodic Schrödinger operators*, arXiv:1712.04700.

[9] J. Li, C. Zhu; *On the reducibility of a class of finitely differentiable quasi-periodic linear systems*, J. Math. Anal. Appl., **413**, 69-83 (2014).

[10] J. You, Q. Zhou; *Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications*, Commun. Math. Phys., **323**, 975-1005(2013).

[11] E. Zehnder; *Generalized implicit function theorems with application to some small divisor problems,* I. Commun. Pure. Math., **XXVIII**, 91-140 (1975).

[12] D. Zhang, J. Xu, X. Xu; *Reducibility of three dimensional skew symmetric system with Liouvillean basic frequencies*, Discrete & Cont, Dyn, Syst., **38(6)**, 2851-2877 (2018).